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Quiver representations [1,2,3], invariant theory and Coxeter groups

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***Abstract*-** We utilized root systems, via correspondence, in other areas of study. In particular, we considered quiver representations[1,2,3], invariant theory and Coxeter groups.

I. Preliminaries

A reflection is natural geometric concept. It is a linear transformation of Euclidean space that fixes a hyper plane H points wise and sends its orthogonal vectors to their opposite with respect to H.

Here we will introduce Coxeter systems and Weyl group and their classifications. While the finite reflection groups have a special type of root system. To clarify that we consider the following concepts:

(i) Let $\mathfrak{g}$ be Lie algebra then the Killing form on $\mathfrak{g} \times \mathfrak{g}$ is defined by

$$B(X,Y) = -Tr(ad\,X \circ ad\,Y) \in R$$

(ii) The Lie algebra $\mathfrak{g}$ is semisimple if and only if B is non- degenerate.
and

(iii) $h \in \mathfrak{h}$ where $\mathfrak{h}$ is Cartan sub algebras, then we can define abstract root systems as follows**:**

II. Abstract Root systems

If B is non – degenerate on h , so there is an induced isomorphism $: h \longrightarrow h^*$. by definition, $< s(h), h' >= B(h\,,h')$
Let's calculate

$$< sH_\beta, H_\alpha > = B(H_\beta, H_\alpha) = B(H_\alpha, H_\beta) \qquad (B \text{ Symmetric})$$

$$= B(H_\alpha, [X_\beta, Y_\beta]) = B([H_\alpha, X_\beta], Y_\beta) \qquad (B \text{ invariant})$$

$$= B(X_\beta, Y_\beta)B(H_\alpha)$$

$$= \tfrac{1}{2}B([H_\beta, X_\beta], Y_\beta)\beta(H_\alpha) \qquad (2X_\beta = [H_\beta, X_\beta])$$

$$= \tfrac{1}{2}B(H_\beta, H_\beta)\beta(H_\alpha) \qquad (B \text{ invariant})$$

Thus, we have that $s(H_\beta) = \frac{(H_\beta\,,H_\beta)}{2}\beta$, also compute

$$(\alpha, \beta) =< \alpha, s^{-1}\beta > = \alpha\left(\frac{2H_\beta}{B(H_\beta,H_\beta)}\right) = \frac{2\alpha(H_\beta)}{B(H_\beta,H_\beta)} \quad \ldots\ldots\ldots\ldots\ldots (1)$$

Inparticular, letting $\alpha = \beta$, we get $s(H_\beta) = \frac{2\beta}{(\beta,\beta)}$. this is sometimes called the co-root of , and denoted $\breve{\beta}$. then we can use (1) to rewrite this fact

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For $\alpha, \beta \in \Delta$, $\frac{2(\alpha,\beta)}{(\beta,\beta)} \in Z$ and $\alpha - \frac{2(\alpha,\beta)}{(\beta,\beta)} \beta \in \Delta \Longrightarrow \Delta \subseteq h^*$ (set of roots)

Now we can define $r_\beta : h^* \longrightarrow h^*$ by $r_\beta(r) = x - \frac{2(\alpha,\beta)}{(\beta,\beta)} \beta$. this is the reflection through the plane orthogonal to β in h^*. The group generated by the r_β for $\beta \in \Delta$ **is a Coxeter group** .

Properties of root system :

Basic properties of the root decomposition are :

1. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$

2. $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ if $\alpha + \beta \neq 0$

3. $B|_{\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}}$ is non – degenerate

4. $B|_h$ is non – degenerate

Definition (2.1) :

An abstract reduced root system is a finite set $\Delta \subseteq R^n \setminus [0]$ which satisfies :

1. Δ spans $\mathbb{R}^n$

2. If $\alpha, \beta \in \Delta$ then $\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$ and $r_\beta(\Delta) = \Delta$

(i.e. $\alpha, \beta \in \Delta \Longrightarrow r_\beta(\alpha) \in \Delta$, with $\alpha - r_\beta(\alpha) \in \mathbb{Z}_\beta$) and

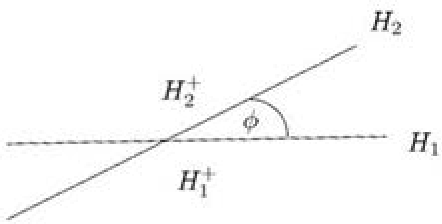
3. If $\alpha, k\alpha \in \Delta$ then $k = \pm 1$ (this is the reduced part), the number n is called the rank of Δ.

Notice:

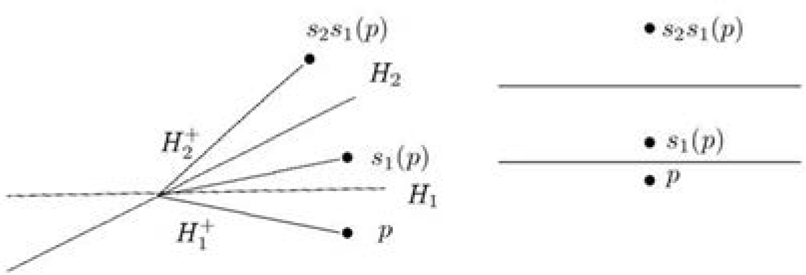
That given root system is $\Delta_1 \subset \mathbb{R}^n$, and $\Delta_2 \subset \mathbb{R}^m$, we get that $\Delta_1 \coprod \Delta_2 \subset \mathbb{R}^n \oplus \mathbb{R}^m$ is root system .

III. Reflection Groups

Suppose H_1 and H_2 are two mirror, H_1^+, H_2^+ is two half planes where $H_i^\pm$ is disjoint union of two half planes ,define the angle $H_1^+ \cap H_2^+$ with measure $\phi = \angle(H_1^+ \cap H_2^+)$, Here $\phi = 0 \Leftrightarrow H_1^- \cap H_2^- = \emptyset$.



Let $s_1 = r_{H_1}$, $s_2 = r_{H_2}$. The composition $s_1 s_2$ is the counterclockwise rotations about the angle 2ϕ if $\phi \neq 0$ and a translation if $\phi = 0$.



G is group generated by the two reflections s_1, s_2.

Definition (3.1):

A reflection group in a space of constant curvature is a discrete group of motions of X^n generated by reflection.

Theorem (3.2):
Let Γ be a reflection group in X^n. There exists a convex polytope $P(\Gamma) = \cap_{i\in I} H_i^-$ such that :
i. P is a fundamental domain for the action of Γ in X^n;
ii. The angle between any two half spaces H_i^-, H_j^- is equal to zero or π/m_{ij} for some positive integer m_{ij} unless the angle is divergent.
iii. Γ is generated by reflections $r_{H_i}, i \in I$.

Definition (3.3):
A finite reflection group is a pair (G, V) where V is Euclidean space, G is a finite subgroup of $O(V)$ and $G = \langle \{S_x : S_x \in G\} \rangle$ generated by all reflections in G.

Generation means : $G \supseteq X$, then X genertaes G if $G = \langle X \rangle$ where X is defined as one of the following equivalents definitions:
i. $\langle X \rangle = \bigcap_{G \geq H \supset X} H$ (semantic)
ii. $\langle X \rangle = \{1\} \cup \left\{a_1^{\pm}, a_2^{\pm}, \dots, a_n^{\pm} : a_i \in X\right\}$ (syntactic)

Equivalence **:**

$(G_1, V_1) \sim (G_2, V_2)$ if there is an isometry $\varphi : V_1 \to V_2$ s.t. $\underbrace{\{\varphi T \varphi^{-1} : T \in G_1\}}_{\varphi G_1 \varphi^{-1}}$

Example (3.4):
$\mathbb{F} = \mathbb{Z}/_{2\mathbb{Z}} = \{0,1\}$ is field of two elements. Consider action of S_n on $\mathbb{F}^n$, $\varepsilon_1, \dots, \varepsilon_n$ basis of $\mathbb{F}^n$.
$\forall \sigma \in S_n : t_\sigma(\varepsilon_i) = \varepsilon_{\sigma(i)}$

Consider semi direct product $S_n \ltimes \mathbb{F}^n = S_n \times \mathbb{F}^n$, the product is
$(\sigma, a).(\tau, b) = (\sigma\tau, t_{\tau-1}(a) + b)$

$S_n \ltimes \mathbb{F}^n$ acts on $\mathbb{R}^n$:
$T_{(\sigma,\alpha)} : e_i \longmapsto (-1)^{a_i}.e_{\sigma_i}$
Let us check that this is the action of $S_n \ltimes \mathbb{F}^n$:

$$
\begin{aligned}
T_{(1,\alpha)}\left(T_{(\tau,0)}(e_i)\right) &= T_{(1,\alpha)}\left(e_{\tau(i)}\right) \\
&= (-1)^{a_{\tau(i)}} e_{\tau(i)} \\
&= (-1)^{[t_\tau - 1(a)]_i} e_{\tau(i)} \\
&= T_{(\tau, t_\tau - 1(a))}(e_i)
\end{aligned}
$$

$B_n = (S_n \ltimes \mathbb{F}^n, \mathbb{R}^n)$
It is reflection since $S_n \ltimes \mathbb{F}^n = \langle ((i,j),0), (1,\varepsilon_i) \rangle$ and $T_{((i,j),0)} = S_{e_i - e_j}$, $T_{(1,\varepsilon_i)} = S_{e_i}$. $|B_n| = n!\ 2^n$, $B_2 \sim I_2(4)$

IV. Coxeter Systems

Definition (4.1):
A group W is a Coxeter group if there exists a subset $S \subseteq W$ such that

$W = \langle s \in S | (ss')^{m_{ss'}} = 1 \rangle$

where $m_{ss'} = 1$ and $m_{ss'} \in \{2,3, \dots\} \cup \{\infty\}$ for all $s \neq s'$. The pair (*W*, *S*) is then called a **Coxeter system**.

1. Let (W,S) be a Coxeter system. If W is finite then we say that (W,S) is finite. If $W = W_1 \times W_2$ and $S = S_1 \sqcup S_2$. Where $\emptyset \neq S_i \subseteq W_i$ and (W_i, S_i) is Coxeter system for $= 1,2$, one say that (W,S) is reducible. Other wise (W,S) is irreducible.

2. Let (W,S) be a Coxeter system. The Coxeter graph X assigned to (W,S) is constructed as follow:

(i) The elements of S form the vertices of X;

(ii) given $s, s' \in S$, there is no edge between s and s' if $m_{ss'} = 2$;

(iii) given $s, s' \in S$, there is an edge labelled by $m_{ss'}$ between s and s' if $m_{ss'} \geq 3$

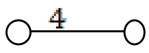
Example (4.3):

Use the definition(4.1) to find the Coxeter system for $\mathbb{F} = \mathbb{Z}/_{2\mathbb{Z}}$

$\mathbb{Z}/_{2\mathbb{Z}} = \langle s_3 | s_3^2 = 1 \rangle$.

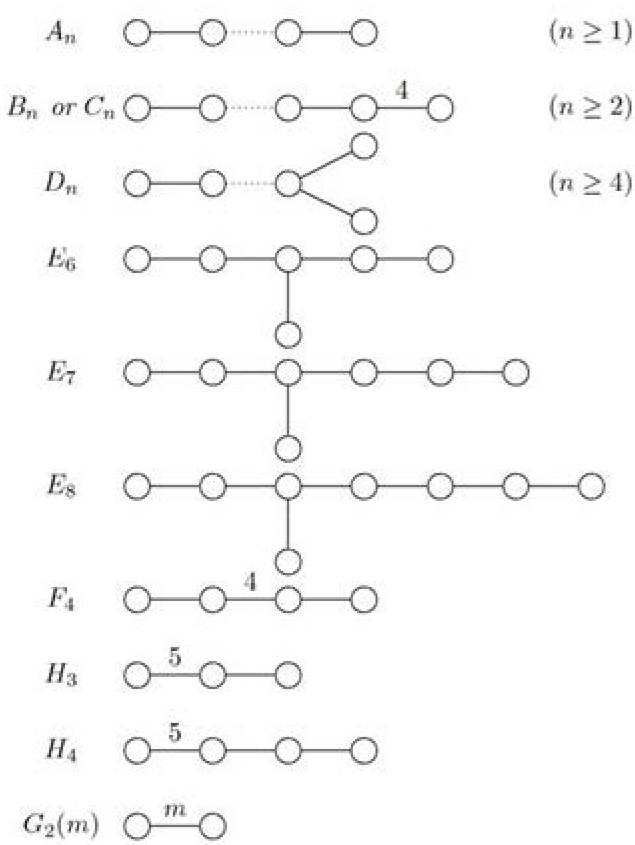
Example (4.4):

The Coxeter graph of B_2 following :



Theorem (4.5):

If (W,S) is a finite irreducible Coxeter system, then its Coxeter graph is one of the following :



Theorem (4.6):

Each of the Coxeter systems represented by the Coxeter graphs $A_1, B_1, \ldots, G_1(m)$ arises from a finite reflection group. Hence the map is surjective and we get a bijection

$$\left\{\begin{matrix}\text{stable isomorphic classes} \\ \text{of finite reflection group}\end{matrix}\right\} \overset{1:1}{\leftrightarrow} \left\{\begin{matrix}\text{isomorphic classes} \\ \text{of finite Coxeter systems}\end{matrix}\right\}$$

V. WEYL GROUPS

Here we will study Weyl groups as special case of finite reflection groups. Indeed they are finite Euclidean reflection groups defined over $\mathbb{Z}$ instead of $\mathbb{R}$.

Definition (5.1):

A lattice of rank l is a free $\mathbb{Z}$- module $\mathcal{L} = \mathbb{Z}^l$. A weyl group is a finite Euclidean reflection group $W \subseteq O(\mathbb{E})$ admitting a W-invariant lattice $\mathcal{L} \subseteq \mathbb{E}$,where

$$\mathbb{E} = \mathcal{L} \otimes_{\mathbb{Z}} \mathbb{R}.$$

Note that for all $\alpha, \beta \in \Delta$ we have the following identity

$$\langle \alpha, \beta \rangle = 2\frac{\|\beta\|}{\|\alpha\|}\cos\theta ,$$

Where $\theta = \theta_{\alpha,\beta}$ is the angle between these vectors. Thus we have

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 4\cos^2\theta$$

Since the only root system Δ is crystallographic, we must have $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 0,1,2,3$ or 4.

Hence the only possibilities for θ are $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}$ and $\frac{5\pi}{6}$.

VI. DYNKIN DIAGRAMS

Definition (6.1):

Now if we change the notaion for Coxeter graphs, we get a Dynkin diagrams. Namely, for Δ an essential crystallographic root system and Σ a fundamental system of Δ, we assign a graph X to Δ as follows :

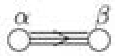
The elements of Σ from the vertices of X.

Given $\alpha \neq \beta \in \Sigma$ and θ the angle between them, we assign $0,1,2$ or 3 edge(s) between α and β by the following rule

$\theta = \frac{\pi}{2}$	No edge
$\theta = \frac{2\pi}{3}$	1 edge
$\theta = \frac{3\pi}{4}$	2 edges
$\theta = \frac{5\pi}{6}$	3 edges

Lemma (6.2):

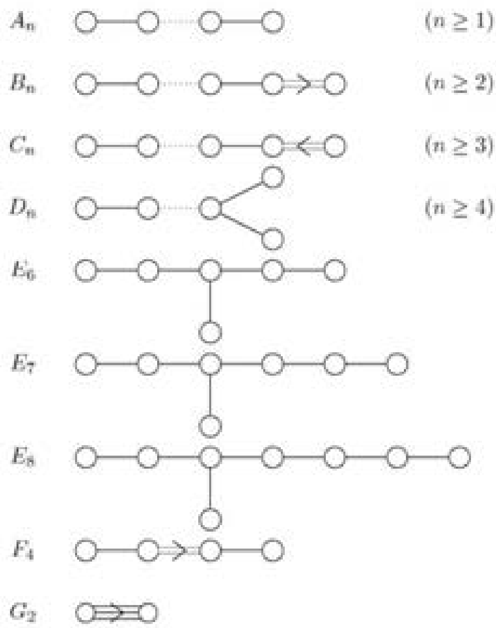
If Δ is irreducible, then at most two root lengths occur in Δ. If two root lengths occur in Δ, we call the roots short and long. We denote it in the Coxeter graph of Δ by an arrow pointing towards the shorter root.i.e., if $\|\alpha\| > \|\beta\|$ we have

or


A Coxeter graph with such arrow is called a Dynkin diagram.

Theorem (6.3):

If Δ is an irreducible essential crystallographic root system, then its Dynkin diagram is one of the following:



Theorem (6.4):

There exists a crystallographic root system having each of $A_1, B_1, \ldots, F_4, G_2$ as its Dynkin diagram.

VII. Conclusion

A finite reflection group with a special type of root system has a Coxeter graph made a Coxeter system that we can represented it by a Dynkin diagram with such an arrow. a group can have more than one Coxeter system.

References

[1] Lecture Note – Reflection Groups – Dr.Dmitriy Rumynin – Anne Lena Winstel – Autumn Term 2010.

[2] Reflection Groups and Rings of Invariants - M´elanie ULDRY- Lausanne, spring semester 2012.

[3] Reflection groups in algebraic geometry – Igor V. Dolgachev – article electronically published on October 26, 2007.

[4] Structure theory of semisimple Lie groups – A.W.Knap -1997.

[5] Introduction to Lie algebras – Alexander Kirillov.Jr – Departement of mathematics, Sunny at Stony Brook, NY 11794, USA.

[6] Cartan matrices with null roots and finite Cartan Matrices – Stephen Berman, Robert Moody & Maria Wonen Burger – University of Saskatchewan – Indiana University – Novemeber 22,1971.

[7] Relaunch.hcm.uni-bonn.de/fileadmin/perrin/chap5.pdf

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