# Utility of irreducible group representations in differential equations (II)

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***Abstract-*** We introduce the concept of a symmetry group of a system of partial differential equations and group-invariant solutions to PDE . Given any system of partial differential equations, it is shown how, in principle, to construct group invariant solutions for any group of transformations by reducing the number of variables in the system. Conversely, every solution of the system can be found using the reduction method with some weak symmetry group.

***Index Terms-*** symmetry group , invariant solution , prolongation.

## I. Introduction

Differential invariants play a central role in a wide variety of problems arising in geometry , differential equations, mathematical physics, and applications,[6].

The construction of explicit solutions to partial differential equations by symmetry reduction dates back to the original work of Sophus Lie,[3] . He demonstrated that for a given system of partial differential equations the Lie algebra of all vector fields (i.e., infinitesimal generators of local one-parameter groups transforming the independent and dependent variables) leaving the system invariant could be straightforwardly found via the solution of a large number of *auxiliary* partial differential equations of an elementary type, the so-called "defining equations" of the group,[4].

## II. Group Invariant Solutions of Differential Equation

2.1 *DEFINITION* Let $\Delta$ be a system of partial differential equations. A strong symmetry group of $\Delta$ is a group of transformations $G$ on the space of independent and dependent variables which has the following two properties:

(a) The elements of $G$ transform solutions of the system to other solutions of the system.

(b) The $G$-invariant solutions of the system are found from a reduced system of differential equations involving a fewer number of independent variables than the original system $\Delta$. (The degree of reduction is determined by the dimension of the orbits of $G$).

A weak symmetry group of the system $\Delta$ is a group of transformations which satisfies the reduction property (b), but no longer transforms solutions to solutions.

Lie Groups of Differential Equations,[1].

In considering Lie groups of point transformations associated to a given differential equation $\Delta$ involving $n$ independent variables $= (x_1,\dots,x_n) \in \mathbb{R}^n$ , and $m$ dependent variables $u = (u^1,\dots,u^m) \in \mathbb{R}^m$, let us write such a group of transformations in the form :

$$x^* = X(x,u;\,a),\quad u^* = U(x,u;\,a)\,. \qquad (2.1)$$

acting on the space $R^{n+m}$ of the variables $(x,u)$. Also, let

$$u = f(x) \equiv \left(f^1(x), f^2(x), \dots, f^m(x)\right) \quad (2.2)$$

be a solution of the equation $\Delta$.

A Lie group of transformations of the form (2.1) admitted by $\Delta$ has the two equivalent properties:

1. a transformation of the group maps any solution of $\Delta$ into another solution of $\Delta$;

2. a transformation of the group leaves $\Delta$ invariant, say, $\Delta$ reads the same in terms of the variables $(x,u)$ and in terms of the transformed variables $(x^*,u^*)$,[2].

*3. Invariant Solutions of Partial Differential Equations*

The function $u = f(x)$, with components $u^{\alpha} = f^{\alpha}(x)$, $(\alpha = 1,2,\ldots,m)$, is said to be an *invariant solution* of $\Delta(x, u^{(1)}, \ldots, u^{(k)}) = 0$ if $u^{\alpha} = f^{\alpha}(x)$ is an invariant surface of (1.1), and is a solution of ($\Delta(x, u^{(1)}, \ldots, u^{(k)}) = 0$), *i.e.*, a solution is invariant if and only if:

$$X(u^{\alpha} - f^{\alpha}(x)) = 0 \;\; \text{for } u^{\alpha} = f^{\alpha}(x), \;\; (\alpha = 1,2,\ldots,m)$$
$$\Delta(x, u, u^{(1)}, \ldots, u^{(k)}) = 0 \qquad (3.1)$$

The equations (3.1) 1, called *invariant surface conditions*, have the form

$$\xi_1(x,u)\frac{\partial u^{\alpha}}{\partial x_1} + \cdots + \xi_n(x,u)\frac{\partial u^{\alpha}}{\partial x_n} = \eta^{\alpha}(x,u), \quad (\alpha = 1,\ldots,m) \qquad (3.2)$$

and are solved by introducing the corresponding characteristic equations:

$$\frac{dx_1}{\xi_1(x,u)} = \cdots = \frac{dx_n}{\xi_n(x,u)} = \frac{du^1}{\eta^1(x,u)} = \cdots = \frac{du^m}{\eta^m(x,u)} \qquad (3.3)$$

This allows to express the solution $u = f(x)$ (that may be given in implicit form if some of the infinitesimals $\xi_i$ depend on $u$) as

$$u^{\alpha} = \psi^{\alpha}(I_1(x,u), \ldots, I_{n-1}(x,u)), (\alpha = 1,2,\ldots,m) \qquad (3.4)$$

by substituting (3.4) into (3.1) 2, a reduced system of differential equations involving *n-1* independent variables (called *similarity variables*) is obtained. The name *similarity variables* is due to the fact that the scaling invariance, *i.e.*, the invariance under similarity transformations, was one of the first examples where this procedure has been used systematically,[2].

To compute point symmetries , we introduce the notion of a vector field. For our purposes , a vector field is a first order differential operator, which can be written,

$$v(f) = \sum_{k=1}^{m} \xi_k(x,u)\frac{\partial f}{\partial x_k} + \Phi(x,u)\frac{\partial f}{\partial u} \qquad (3.5)$$

A slightly more general form of a vector field is used in Lie's theorem and the prolongation formula given below, but it is only vector fields of the form (3.5) which we will need. The vector fields we deal with will all be *right invariant.*

(3.2) *THEOREM* ( Olver ; The General Prolongation Formula). *Let*

$$v = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \Phi_{\alpha}(x,u)\frac{\partial}{\partial u^{\alpha}} \qquad (3.6)$$

*be a vector field defined on an open subset* $M \in X \times U$. *The n-th prolongation of* $v$ *is the vector field*

$$\mathrm{pr}^{(n)}v = v + \sum_{\alpha}^{q}\sum_{J} \Phi_{\alpha}^{J}(x,u)\frac{\partial}{\partial u_J^{\alpha}} \qquad (3.7)$$

*defined on the corresponding jet space* $M^{(n)} \in X \times U^{(n)}$, *the second summation being over all (unordered) multi-indices* $J = (j_1 1, \ldots, j_k)$, *with* $1 \le j_k \le p$, $1 \le k \le n$. *The coefficient functions* $\Phi_{\alpha}^{J}$ *of* $\mathrm{pr}^{(n)}v$ *aregiven by the following formula:*

$$\Phi_{\alpha}^{J}(x,u^{(n)}) = D_J\left(\Phi_{\alpha} - \sum_{i=1}^{p}\xi^i u_i^{\alpha}\right) + \sum_{i=1}^{p}\xi^i u_{J,i}^{\alpha} \qquad (3.8)$$

*where* $u_i^{\alpha} = \frac{\partial u^{\alpha}}{\partial x^i}$, *and* $u_{J,i}^{\alpha} = \frac{\partial u_J^{\alpha}}{\partial x^i}$ , *and* $D_J$ *is the total differentiation operator.* Although the details involved in the construction of Lie's theory of symmetry groups are quite technical, the application of symmetry methods to PDEs is straightforward,[1].

We will consider a single PDE of order $n$in $m$ variables, defined on a simply connected subset $\Omega \times \mathbb{R}^m$. The PDE takes the general form

$\Delta(x, D^{\alpha}u) = 0;$

where $\Delta$ is a differential operator on $\Omega \times \mathbb{R}$,

$$D^{\alpha}u = \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_m}},$$

and $\alpha = (\alpha_1, \dots, \alpha_m)$ is a multi-index.

(3.3) *DEFINITION* . A symmetry group of a system of differential equations is a local group of transformations $G$acting on the independent and dependent variables of the system such that it maps solutions of the equations to other solutions. More precisely, let $\mathcal{H}_{\Delta}$denote the space of all solutions of the system of PDEs
$\Delta_i(x, D^{\alpha}u) = 0 , \quad i = 1,2,\dots,p .$

A symmetry $S$ is a mapping of $H_{\Delta}$ into itself. i.e. $S: \mathcal{H}_{\Delta} \to \mathcal{H}_{\Delta}$.Thusif $u \in \mathcal{H}_{\Delta}$, then we must have $Su \in \mathcal{H}_{\Delta}$.

(3.4) *EXAMPLE* . Consider the one dimensional heat equation $u_t = u_{xx}$. If$u(x,t)$is a solution of the one dimensional heat equation then $u(x+\epsilon, t)$is also a solution, at least for $\epsilon$sufficiently small. This is a symmetry.

As we will see below, the heat equation has more interesting symmetries than this simple example.

(3.5) *DEFINITION* . Let $G$be a group which acts transversally on a manifold$\mathcal{M}$. Let$x \in \mathcal{M}$and let the action of an element $g \in G$ on $M$ be denoted $g.x$. The *orbit* of $x$under $G$is the set

$$\mathcal{O}_x = \{y \in \mathcal{M} \mid y = g.x, g \in G\}. \qquad (3.9)$$

That is, the orbit is the set of all points that $x$is mapped to as $g$variesthrough the whole group.

(3.6) *EXAMPLE* . Consider the group $SO(2)$. If we take $(x, y) \in \mathbb{R}^2$thenthe orbit of $(x, y)$ under $SO(2)$ is the set of points of the form

$$(x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta), \theta \in [0, 2\pi)$$

It is not hard to identify this set of points. Elementary algebra shows that

$$(x \cos\theta - y)^2 + (x \sin\theta + y \cos\theta)^2 = x^2 + y^2.$$

This is clearly the equation of a circle of radius $= \sqrt{x^2 + y^2}$ . So the orbits are circles. There is also a degenerate case, namely when$(x, y) = (0, 0)$. Here the orbit is the single point $(0, 0)$

Notice that the orbit in the first case is a *submanifold* of $\mathbb{R}^2$.This is always the case, at least under suitable technical assumptions. In the case where the orbits are circles , the dimension of the orbit is one , since a circle is a one dimensional manifold.

In general, if $\mathcal{O}$ is an orbit , then the dimension of the orbit is the dimension of $\mathcal{O}$ regarded as a submanifold.

Suppose that we have a PDE $\Delta(x, D^{\alpha}u) = 0$, in $n$ variables. Supposealso that there exists a symmetry group $G$of the PDE and that theorbits of $G$form a submanifold of dimension $< n$. Then the PDE$\Delta(x, D^{\alpha}u) = 0$ can always be reduced under a change of variables to a PDE in *n-p* variables. In the literature it is common to write the reduced equation as $\Delta/G(y, D^{\alpha}v) = 0$, where $y$and $v$are the new variables given by the change of variables .The key to the method is finding invariants of the group action.

Suppose that we have a one parameter group which is generated by a vector field of the form (2.1). To determine the action of $v$ on a function $f$, we can form the Lie series

$$exp(\epsilon v) f(x) = f(x) + \epsilon v\big(f(x)\big) + \tfrac{1}{2}\epsilon^2 v^2\big(f(x)\big) + \dots .$$

If $f$is invariant under the action of $v$ then $f(x) = exp(\epsilon v) f(x)$ whichimplies that $v(f) = 0$.This means that the invariants are found by solving the first order PDE

$$\sum_{i=1}^{n} \xi_i(x, u) \frac{\partial f}{\partial x_i} + \Phi(x, u) \frac{\partial f}{\partial u} = 0 .$$

This can be done by the method of characteristics. We illustrate the procedure by finding invariants for some vector fields.
(3.7) *EXAMPLE* . Let

$$v = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

This generates rotations. To find the invariants we must solve $v(f) = 0$.

By the method of characteristics we see that the general solution of this PDE is $f = G(x^2 + y^2)$,for $G$an arbitrary differential function.

(3.8) *EXAMPLE* . The one dimensional heat equation $u_t = u_{xx}$ has the symmetry

$$v = 4xt\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - (x^2 + 2t)u\frac{\partial}{\partial u}.$$

We find invariants of the group generated by $v$ by solving

$$\frac{dx}{4xt} = \frac{dt}{4t^2} = -\frac{du}{(x^2 + 2t)u}.$$

To demonstrate the method, we will solve this in steps. First, solving

$$\frac{dx}{4xt} = \frac{dt}{4t^2}$$

gives $\ln x = \ln t + C$.Therefore $C = \ln(x/t)$ .

We could take $\eta = \ln(x/t)$,but since we may actually take any function of $\ln(x/t)$ for $\eta$,it makes sense just to write $\eta = e^{\ln(x/t)} = x/t$. *(1)*

Notice that we then have $x = \eta t$. Returning to previous equation we have to solve

$$\frac{dt}{4t^2} = -\frac{du}{(\eta^2 t^2 + 2t)u}$$

Integration leads to

$$\ln u = -\frac{1}{4}\eta^2 t - \frac{1}{2}\ln t + D.$$

Where $D$is the result of combining the constants of integration from both sides of the equation. Since $\eta = x/t$we have

$$D = \ln(\sqrt{t}u) + \frac{x^2}{4t}.$$

This gives us our second invariant. In fact, any function of $D$will be asecond invariant. Let us take $v = e^D$as our second invariant. That is , we set

$$v = \sqrt{t}e^{-x^2/4t}u \qquad (2)$$

to be the second invariant. Notice that the two sets of invariants wehave obtained are *functionally independent.* Two invariants $\eta$and $\xi$are*functionally dependent* if there exists a continuous function $F$such that$\eta = F(\xi)$.If no such relationship exists, they are said to be functionallyindependent. In general, for a vector field with three variables, there will be two functionally independent sets of invariants.

We use the invariants to rewrite the PDE. We saw that $\eta = x^2 + y^2$is an invariant of the rotation group $SO(2)$ in the plane. The Laplaceequation $\Delta u = 0$ has $SO(2)$ as a group of symmetries.

Let $r = x^2 + y^2$ be our invariant. We look for a solution of the Laplace equation of the form $u(x, y) = U(x^2 + y^2) = U(r)$.Then bythe chain rule Laplace's equation in the plane therefore becomes

$$\Delta u = 4\frac{dU}{dr} + 4r\frac{d^2U}{dr^2} = 0.$$

This ODE is called the *reduced equation*, because we have reduced a PDE in two variables to an ODE.

We may solve this to obtain $U(r) = A\ln r + D$, where $D$ is a constant of integration. This is the family of solutions of the Laplace equationinvariant under rotations. If we take $D = 0$ and $A = 1/4\pi$we obtainthe solution

$$U(r) = \frac{1}{4\pi}\ln r = \frac{1}{4\pi}\ln(x^2 + y^2).$$

which is the fundamental solution of the two dimensional Laplace equation.

What would happen if we chose a different invariant for the change of variables? We could equally have picked $r = \sqrt{x^2 + y^2}$ for the Laplace equation. Doing this, we would obviously arrive at a different ODE. However the ODE which we arrived at would be equivalent to the previous one under the simple change of variable s $\rightarrow \sqrt{r}$ .This is a special case of a more general situation. If $\eta$is an invariant and$\xi = f(\eta)$ is another invariant, then the reduced equations we obtainby using $\eta$and $\xi$respectively as changes of variables, will always beequivalent under the change of variables $\eta \rightarrow f(\eta)$.

Judicious choice of invariants can lead to easier forms of the reduced equation. For example, the so called heat equation on the $ax + b$groupcan be written as

$$\frac{\partial u}{\partial t} = y^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + y\frac{\partial u}{\partial y}.$$

This has scaling symmetries $(x, y, t) \rightarrow (\lambda x, \lambda y, t), \lambda > 0$.If we makethe obvious choice for an invariant $\xi = x/y$then the reduced equationis

$$\frac{\partial u}{\partial t} = (1 + \xi^2)\frac{\partial^2 u}{\partial \xi^2} + 2\xi\frac{\partial u}{\partial \xi}.$$

However, the less obvious choice $\xi = \sinh^{-1}(x/y)$ leads to the reduced PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \xi^2}.$$

This is just the one dimensional heat equation. So, some experimentation may be needed to obtain an optimal form of the reduced PDE.

Having illustrated the general procedure, let us turn to the heat equation.

(3.8) EXAMPLE . Let us find the group invariant solutions for the heat equation $u_t = u_{xx}$, where the group action is generated by the vector field

$v = 4xt\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - (x^2 + 2t)u\frac{\partial}{\partial u}$. We found two invariants, namely $\eta = x/t$ and $v = \sqrt{t}e^{-x^2/4t}u$. Let ourchange of variables be $y = x/t$ ,$v = \sqrt{t}e^{-x^2/4t}u$ .

Applying the chain rule we have

$$u_t = \left(\frac{(-2t + x^2)v(y) - 4xv^{'}(y)}{4t^{\frac{5}{2}}}\right) e^{-x^2/4t}$$

Turning to the *x* derivatives gives

$$u_{xx} = \left(\frac{(-2t + x^2)v(y) - 4xv^{'}(y) + 4xv^{''}(y)}{4t^{\frac{5}{2}}}\right) e^{-x^2/4t}$$

So using our expressions for $u_t$and $u_{xx}$, the heat equation becomes$v^{''}(y) = 0$.

The general solution is just $v(y) = Ay + B$. Hence the group invariant solutions are of the form

$$u(x,t) = \frac{1}{\sqrt{t}} e^{-x^2/4t}\left(A\frac{x}{t} + B\right).$$

Taking $A = 0\,, B = \frac{1}{\sqrt{4\pi}}$ will give the fundamental solution of the heat equation,[5].

## III. Conclusion

Group invariant solutions of a system of differential equations are very useful in classifying the solutions of the differential equations . Following the previous results in the previous paper , we my use the adjoin maps to carry on classification skew . In the present paper we show how to find group invariant solutions for some cases .

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