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WEAKLY PRIME ELEMENTS IN LATTICE MODULES

C S MANJAREKAR AND U N KANDALE

Abstract. As a generalization of the notion of prime element and semiprime element, we introduce the notion of weakly prime element and weakly semiprime element in lattice modules. Some characterization of weakly prime and weakly semiprime elements are obtained. Throughout this paper, L will be a lattice domain.

Keywords:
Multiplicative lattice, prime element, weakly prime element, semiprime element, weakly semiprime element.

1. Introduction

Several years ago, R.P.Dilworth [1] began the study of abstract commutative ideal theory. Where he introduced the notion of multiplicative lattices. Multiplicative lattice is a complete lattice provided with commutative, associative and join distributive multiplication for which largest element 1 acts as a multiplicative identity. R. P. Dilworth also defined the notion of principal element as a generalization of the of principal ideal and defined a Noether lattice. Then J. A. Johnson and E. W. Johnson [3] introduced a concept of Noetherian lattice modules and then many ideas of Dilworth were extended. Let L be a multiplicative lattice. A lattice module over L or simply a lattice module is defined to be a complete lattice M with multiplication $L \times M \to M$ satisfying,

(1) $(\underset{\alpha}{\vee} a_\alpha)A = \underset{\alpha}{\vee} a_\alpha A \quad \forall\ a_\alpha \in L,\ A \in M$
(2) $a(\underset{\alpha}{\vee} A_\alpha) = \underset{\alpha}{\vee} aA_\alpha \quad \forall\ a \in L,\ A_\alpha \in M$
(3) $(ab)A = a(bA) \quad \forall\ a, b \in L,\ A \in M$
(4) $IA = A \quad \forall A \in M$
(5) $0A = 0_M \quad \forall A \in M \quad where \quad 0_M = glb(M)$

Elements of L will generally be denoted by a,b,c,........ except that the least element of L will be denoted by O and the greatest element of L will be denoted by I. The elements of M will generally be denoted by A,B,C,...... except that the least element and greatest element of M will be denoted by

O_M and I_M. Here after L will be a multiplicative lattice and M will be a lattice module over L. As in case of commutative rings, there are residuation operations in lattice module, for $a, b \in L \ \ and \ \ A, B \in M$,

a:b is the greatest element c in L such that, $cb \leqslant a$,

A:b is the greatest element C in M such that $bC \leqslant A$ and,

A:B is the greatest element a in L such that $aB \leqslant A$.

An element A in M is called weak meet principal if

$(B : A)A = B \wedge A, \ \ \forall B \in M$,

A is called as weak join principal if

$(bA : A) = b \vee (O : A) \ \ \forall b \in L$

and A is called weak principal if A is both weak meet and weak join principal. An element B in M is called as multiplication element if for every element $D \leqslant B$, there exists an element b of L such that D = bB. We note that, A is weak meet principal element of M if and only if A is a multiplication element. A proper element p of L is called prime element if $ab \leqslant p$ implies $a \leqslant p$ or $b \leqslant p$, for $a, b \in L$. A proper element p of L is called primary element if $ab \leqslant p$ implies $a \leqslant p$ or $b^n \leqslant p$, for some $n \in Z_+$.

An element a of L is called as zero divisor if $\exists \ 0 \neq b \in L$ such that ab = 0, if L has no divisors of zero then L will be called a lattice domain or simply a domain. A proper element m of L is said to be a maximal element if $m \nleqslant a$ for any proper element a of L. An element a of L is called compact if $a \leqslant \vee X, \ \ X \subseteq L$ implies the existence of a finite number of elements $X_1, X_2, X_3, \ldots ....X_n...$, of L such that, $a \leqslant X_1 \vee X_2 \vee ..... \vee X_n$.

Prime elements in Lattice modules are studied by E A Khouja [4]. For other definitions and properties in lattice modules one can refer J A Johnson [2].

2. Weakly prime element and weakly semi prime elements

We introduce the concept of weakly prime element and weakly semi prime element in lattice modules.

Definition 2.1. *An element $N \neq I_M$ of a lattice module M is called weakly prime element if whenever $0 \neq aA \leqslant N$ where $a \in L$, $A \in M$ implies either $A \leqslant N$ or $a \leqslant (N : I_M)$*

Example 2.1. *Let M denote a cyclic Z-module $\frac{Z}{6Z}$ and let $L = L(Z)$ denote the set of all ideals of Z and $L(\frac{Z}{6Z})$ denote the set of all submodules of $\frac{Z}{6Z}$. Then $L(\frac{Z}{6Z})$ is a lattice module over L. Let $N = \{0\}$. Then N is a weakly prime element of M.*

Definition 2.2. *An element $N \neq I_M$ of a lattice module M is called weakly semiprime element if $\forall\ a, b \in L$ such that $0 \neq abI_M \leqslant N$ implies $aI_M \leqslant N$ or $bI_M \leqslant N$.*

The next result gives the relationship between weakly prime element and weakly semiprime element.

Theorem 2.1. *Let M be a L-module. Then every weakly prime element of M is weakly semiprime element.*

Proof. Let N be a weakly prime element of a lattice module M. Suppose, $0 \neq abI_M \leqslant N$ where $a, b \in L$. Then either $aI_M \leqslant N$ or $b \leqslant (N : I_M)$. This implies that, either $aI_M \leqslant N$ or $bI_M \leqslant N$. So N is a weakly semiprime element. □

Theorem 2.2. *Let L be a lattice domain and $N \neq I_M$ be an element of lattice module M. Then the following statements are equivalent,*

(1) *N is a weakly semiprime element.*
(2) *For any two elements A and B of M if*
  $(A : I_M)(B : I_M) \leqslant (N : I_M)$ *then* $(A : I_M) \leqslant (N : I_M)$
  or
  $(B : I_M) \leqslant (N : I_M)$ *where* $AB \neq 0$

Proof. $1 \Rightarrow 2$ Suppose that, $(A : I_M)(B : I_M) \leqslant (N : I_M)$ and $(B : I_M) \nleqslant (N : I_M)$. Then $\exists$ an element $0 \neq b \in L$ such that $b \leqslant (B : I_M)$ and $b \nleqslant (N : I_M)$. Let $0 \neq a \leqslant (A : I_M)$. Then $0 \neq ab \leqslant (A : I_M)(B : I_M) \leqslant (N : I_M)$. Therefore, $0 \neq abI_M \leqslant N$.

Since N is weakly semiprime element and $b \nleqslant (N : I_M)$, it follows that, $a \leqslant (N : I_M)$and the proof follows.

$2 \Rightarrow 1$ Suppose that, $0 \neq abI_M \leqslant N$, then $(abI_M : I_M) \leqslant (N : I_M)$.
But, $(aI_M : I_M)(bI_M : I_M) \leqslant (abI_M : I_M)$.
So, either $(aI_M : I_M) \leqslant (N : I_M)$ or $(bI_M : I_M) \leqslant (N : I_M)$.
But, $a \leqslant (aI_M : I_M)$ and $b \leqslant (bI_M : I_M)$.
Thus, either $aI_M \leqslant N$ or $bI_M \leqslant N$ and N is a weakly semiprime element.
□

Theorem 2.3. *Let M be a L-module in which greatest element I_M is multiplication element and let $N \neq I_M$ be an element of M. Then N is weakly prime element if and only if it is weakly semiprime element.*

Proof. Suppose , N is weakly semiprime element and let $0 \neq aX \leqslant N$ where $X \in M$ and $a \in L$.

Therefore, $(aX : I_M) \leqslant (N : I_M)$.
However, $(aI_M : I_M)(X : I_M) \leqslant (aX : I_M)$ which implies that, $(aI_M : I_M)(X : I_M) \leqslant (N : I_M)$.
Using the theorem (2.4), we get either $(aI_M : I_M) \leqslant (N : I_M)$ or $(X : I_M) \leqslant (N : I_M)$.
Hence either$(aI_M : I_M)I_M \leqslant (N : I_M)I_M$ or $(X : I_M)I_M \leqslant (N : I_M)I_M$.
Since, I_M is multiplication element, I_M is weak meet principal (by remark 1.1 in [4]).
By definition of weak meet principal, $aI_M \wedge I_M \leqslant N \wedge I_M$ or $X \wedge I_M \leqslant N \wedge I_M$. Here $aI_M \leqslant N$ or $X \leqslant N$.

Therefore, N is weakly prime element of M. The converse follows from the theorem (2.4). □

Theorem 2.4. *Let $N \neq I_M$ be an element of a L-module M, then the following statements are equivalent,*

(1) *N is a weakly prime element of M.*
(2) *For every element B of M such that $B > N$, we have $(N : B) = (N : I_M)$.*
(3) $(N : X) = (N : I_M)$*, for every element X of M such that $X \nleqslant N$.*
(4) $N = (N : a)$*, for every element a of L such that $(N : I_M) < a$.*

Proof. (1) ⇒ (2)

It is obvious that, $(N : I_M) \leqslant (N : B)$. We must show that, $(N : B) \leqslant (N : I_M)$. Let $0 \neq a \leqslant (N : B)$ then $0 \neq aB \leqslant N$. Since N is a weakly prime element and $B \nleqslant N$, it follows that $a \leqslant (N : I_M)$ which proves that, $(N : B) \leqslant (N : I_M)$ and we have, $(N : B) = (N : I_M)$.

(2) ⇒ (3)

Let X be an element of M such that $X \nleqslant N$ and let $Y = N \vee X$. We have, $(N : Y) = N : (N \vee X) = (N : X)$. But, $(N : Y) = (N : I_M)$ by (2) and hence, $(N : I_M) = (N : X)$ for every $X \nleqslant N$.

(3) ⇒ (4)

Let a be an element of L such that $a > (N : I_M)$. It is obvious that, $N \leqslant (N : a)$. On the other hand, let X be an element of M such that $X \leqslant (N : a)$. Then $aX \leqslant N$ and hence $a \leqslant (N : X)$. Suppose, $X \nleqslant N$. Then by (3), $(N : X) = (N : I_M)$ and hence $a \leqslant (N : I_M)$ a contradiction.

Thus, $X \leqslant N$ and $N = (N : a)$.

(4) ⇒ (1)

Let, $0 \neq rX \leqslant N$ and $r \nleqslant (N : I_M)$. Define, $a = (N : I_M) \vee r$. Then $aX = (N : I_M)X \vee rX \leqslant N$. Hence, $X \leqslant (N : a) = N$. Consequently, N is a weakly prime element. □

Theorem 2.5. *Let M be a L-module. Then every maximal element of M is weakly prime element.*

Proof. Since maximal element is prime and prime implies weakly prime, the result follows. □

Theorem 2.6. *Let N be a weakly prime element of L Module M and let $0 \neq a$ be an element of L. Then either $(N : a) = N$ or $(N : a) = I_M$.*

Proof. Since, $0 \neq a(N : a) \leqslant N$ by definition of weakly prime element, we have $a \leqslant (N : I_M)$ or $(N : a) \leqslant N$. Consequently, either $(N : a) = I_M$ *or* $(N : a) = N$. □

Theorem 2.7. *Let $\{N_i\} i \in I$ be a chain (descending or ascending) of weakly prime element of a L module M then,*

(1) $\underset{i \in I}{\wedge} N_i$ *is weakly prime element of M.*

(2) *If the greatest element I_M of M is compact then $\underset{i \in I}{\vee} N_i$ is weakly prime element of M.*

Proof. (1) Let $N_1 \leqslant N_2 \leqslant N_3 \leqslant ...... \leqslant N_i \leqslant ......$ be an ascending chain of elements of M. It is clear that, $\underset{i \in I}{\wedge} N_i \neq I_M$. Let $0 \neq rX \leqslant \underset{i \in I}{\wedge} N_i$ for $r \in L, X \in M$ and let $X \nleqslant \underset{i \in I}{\wedge} N_i$. Thus, $X \nleqslant N_j$, for some $j \in I$. Consequently, $r \leqslant (N_j : I_M)$. Next, let $N_i \neq N_j$. Then either $N_i < N_j$ or $N_j < N_i$. If $N_i < N_j, X \nleqslant N_i$ and $0 \neq rX \leqslant N_i$, so we have, $r \leqslant (N_i : I_M)$. Let $N_j < N_i$ and hence $r \leqslant (N_j : I_M) \leqslant (N_i : I_M)$. Thus, $r \leqslant \underset{i \in L}{\wedge} (N_i : I_M) = (\underset{i \in I}{\wedge} N_i) : I_M$ which proves that $\underset{i \in I}{\wedge} N_i$ is a weakly prime element of M.

(2) Note that, $\underset{i \in I}{\vee} N_i \neq I_M$, since I_M is compact. Now, let $0 \neq rX \leqslant \underset{i \in I}{\vee} N_i$ and $X \nleqslant \underset{i \in I}{\vee} N_i$. Then $0 \neq rX \leqslant N_j$ for some $j \in I$ but $X \nleqslant N_j$. Accordingly, $r \leqslant (N_j : I_M) \leqslant ((\underset{i \in I}{\vee} N_i) : I_M)$ which proves that $\underset{i \in I}{\vee} N_i$ is weakly prime element of M.

The proof for descending chain in analogous.

□

Theorem 2.8. *Let M be an L module in which the greatest element I_M is weak principal and let a be an element of L. Then aI_M is weakly prime element of M if and only if $a \vee (0 : I_M)$ is weakly prime element of L.*

Proof. By proposition (2.1) in [4], we know that, maximal element implies prime element and prime element implies weakly prime element. We have, $[a \vee (0 : I_M), I_L] = [(aI_M : I_M), I_L] \cong [aI_M, I_M]$, by lemma (2.1) in [4], which proves that, aI_M is a weakly prime element of M if and only if $a \vee (0 : I_M)$ is weakly prime element of L. □

Theorem 2.9. *Let N be an element of an L-module M. If N is weakly prime element of M then* $(N : I_M)$ *is weakly prime element of L where L is a lattice domain.*

Proof. Since, N is a weakly prime element of M, $N \neq I_M$ and hence, $(N : I_M) \neq I$. Let, $0 \neq ab$ be an element of L such that $ab \leqslant (N : I_M)$ and suppose that , $b \nleqslant (N : I_M)$. This means that, $0 \neq abI_M \leqslant N$ and $bI_M \nleqslant N$ but N is a weakly prime element. Therefore, $a \leqslant (N : I_M)$ which implies that, $(N : I_M)$ is a weakly prime element of L. □

Theorem 2.10. *Let M be L-Module in which the greatest element* I_M *is weak principal and let N be an element of M. Then N is weakly prime element of M if and only if* $(N : I_M)$ *is weakly prime element of L.*

Proof. Since, I_M is weak principal, $[(N : I_M), I_L] \cong [N, I_M]$, by lemma (2.1) in [4].

Thus, N is a weakly prime element of M if and only if $(N : I_M)$ is a weakly prime element of L. □

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Prof C S Manjarekar
Department of Mathematics
Shivaji University
Kolhapur
csmanjrekar@yahoo.co.in
+91 9373 5615 30 / +91 9611 6879 25

U N Kandale
General Engineering Department
Sharad Institute of Technology - College of Engineering
Yadrav
ujwalabiraje@gmail.com
+91 8421 360061 / +91 9 6655 9 2207