ISSN 2250-3153

An Explicit Formulation of Franklin Fresnelets

M P Sivaramakrishnan

Department of Mathematics, Government Brennen College
Thalassery, INDIA. Pin 670106
e-mail:sivanmp@gmail.com

Abstract- The *Fresnel* transform is applied on an ideal wavelet basis for $L^2(\mathbb{R})$, the B-splines [1], which results into the F-splines.In digital holography the image recorded on a CCD has to be reconstructed. Though the underlying idea of *Fresnelets* are given in [2], a detailed mathematical derivation is provided here. As a special instance of *Fresnelets*, the *Franklin Fresnelets* are considered. To meet the requirements of retrieving the image,the *Fresnelets* are more effective than the *wavelets* derived from the B-spline scaling functions.

Index Terms- *F-splines,Scaliing functions,Wavelets, Fresnelets, Franklin Fresnelets*.

Introduction

To reconstruct a complex wave near an object the *Fresnel* diffraction integral is applied. Since the process is involving lenseless CCD device, standard wavelets are inefficient to provide the detail at the edges of the image which are recorded poor when applied to hologram [2]. Though *Gabor Wavlets* are good, the excellent choice is B-spline due to several factors.

When the *Fresnel* transform is implemented on a B-spline, we get an F-spline. The B-splines are the scaling functions of a *Multi Resolution Analysis*[3]. So do the F-spline except for the fact that the two scale relation for F-spline which involves the parameter τ slightly differ from that of B-spline.

In section (1) we have some definitions and theorems and in section (2) we prove our main results.

1. *F*-splines

We begin with some preliminary notions.

Definition 1. [4] *The cardinal B-spline* $\beta^{\mathbf{n}}$ *of order* n *is a function in* $C^{n-1}(\mathbb{R})$ *equally spaced with integer knots and are polynomials of degree* n *in the interval* $[2^{-m}k, 2^{-m}(k+1)]$.

It has compact support in $[0,n]$ and V_0 = span $\{\beta^n(x-k): k\in\mathbb{Z}\}$ is a subspace of $L^2(\mathbb{R})$. By writing V_j = span$\{\beta^n(2^j x-k): k\in\mathbb{Z}\}$, the multi resolution analysis in $L^2(\mathbb{R})$ is defined as the following.

Definition 2. *[Multi Resolution Analysis]* [1]

(i) $V_j \subseteq V_{j+1}$, $\bigcap_{j\in Z} V_j = \{0\}$ and $\overline{\bigcup_{j\in Z} V_j} = L^2(\mathbb{R})$

(ii) $f(x)\in V_j \Leftrightarrow f(2x)\in V_{j+1} \quad \forall f\in L^2(\mathbb{R})$ (scale invariance)

(iii) $f(x)\in V_0 \Leftrightarrow f(x-k)\in V_0 \quad \forall f\in L^2(\mathbb{R})$ (shift invariance)

(iv) The set $\{\beta^n(x-k): k\in\mathbb{Z}\}$ is an orthonormal basis for V_0

Proposition 1.1. [1] *The B-spline of order* 1 *is defined as* $\beta^1(x) = \chi_{[0,1]}(x)$, χ *is the characteristic function. Inductively,* $\beta^n(x) = \beta*\beta*\cdots*\beta(x)$ *(n factors). Then* $\beta^n(x) = \sum_{k=0}^{n+1}(-1)^k \binom{n+1}{k}\frac{(x-k)_+^n}{n!}$ *where* $x_+^n = (0,x)^n$

In the Fourier domain $\hat{\beta}^n(w) = (\hat{\beta}^1(w))^n$. We have $\hat{\beta}^1(w) = e^{-\frac{iw}{2}} \dfrac{\sin\frac{w}{2}}{\frac{w}{2}} = e^{-\frac{iw}{2}} sinc(\frac{w}{2})$.

Therefore $|\hat{\beta}^n(w)| = |\hat{\beta}^1(w)|^2 = |sinc(w/2)|^n$ where $sincx = \begin{cases} \dfrac{\sin x}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$.

Definition 3. *[2] The Fresnel Transform of* $f \in L^2(\mathbb{R})$ *is a unitary map* $\tilde{f}_\tau : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *defined by* $\tilde{f}_\tau(x) = f * k_\tau(x)$, *where the Fresnel operator* $k_\tau(x) = \frac{1}{\tau} e^{i\pi(\frac{x}{\tau})^2}$, $\tau > 0$.

Properties

The *Fresnel Transform* satisfies the following properties :

(i) $\tilde{f}(\cdot - x_0)_\tau(x) = \tilde{f}_\tau(x - x_0)$ (shift invariance)

(ii) $f^*(x) = (\tilde{f}_\tau\tilde{)}_\tau(x)$ (Duality)

(iii) $(\tilde{f}(\frac{\cdot}{s}))_\tau(x) = \tilde{f}_{\frac{\tau}{s}}$, $s \neq 0$ (Dilation);

(iv) $\langle f, g \rangle = \langle \tilde{f}_\tau, \tilde{g}_\tau \rangle$ and $\|f\| = \|\tilde{f}_\tau\|$ (Parseval Relation and Plancheral property)

(v) In the Fourier domain $\hat{k}_\tau(w) = e^{i\frac{\pi}{4}} e^{-i\pi(\tau w)^2}$

Definition 4. *The definition of* F *-spline goes along with that of* B *-spline. It is expressed as the Fresnel Transform of the* B *-spline* $\beta^n(x)$, *that is the* F *-spline* $\tilde{\beta}^n_\tau(x) = \beta^n * k_\tau(x)$

Theorem 1. [2] *The* F *-spline satisfy all the three requirements of a valid scaling function viz.* (i) *It is a Riesz basis,* (ii) *It has partition of unity and* (iii) *It has two scale relation*

Remark 1. *Due to the presence of the parameter* τ *the two scale relation has the modified form* $\tilde{\beta}^n_\tau(x) = \sum_{k \in Z} h(k) \tilde{\beta}^n_{2\tau}(2x - k)$.

2. Fresnelets

In this section we make a concrete construction of Fresnelets.

Definition 5. *Fresnelets are the wavelets derived out of the scaling functions F-splines. To develop this we derive the Franklin wavelets, the wavelets associated with the B-spline of order 2.*

Lemma 1. [4] *The set* $\{\phi(x-k)\}$ *is an orthonormal system in* $L^2(\mathbb{R}) \Leftrightarrow \sum_{k \in Z} |\hat{\phi}(w + 2k\pi)|^2 = 1$

Theorem 2. [4] *A function* ϕ *is a scaling function in* $L^2(\mathbb{R})$ *if and only if*

$$\hat{\phi}(w) = \left(\frac{\sin(\frac{w}{2})}{\frac{w}{2}} \right)^2 m_\phi(w)$$

where m_ϕ *is a* 2π *periodic function in* $L^2([0, 2\pi))$.

Proposition 2.1. *The* B *-spline* β^2 *satisfy the identity*

$$\sum_{k \in Z} |\hat{\beta}^2(w + 2\pi k)|^2 = 1 - \frac{2}{3}\sin^2(\frac{w}{2})$$

We know $|\hat{\beta}^n(w+2\pi k)|^2 = \dfrac{\sin^{2n}(\frac{w}{2})}{(\frac{w}{2}+\pi k)^{2n}} \quad \Theta \quad |\hat{\beta}^n_{\psi}(w)| = \sin c w$

Replace w *by* $2w$ *and sum over all* $k \in Z$

$$\sum_{k\in Z}|\hat{\beta}^n(w+2\pi k)|^2 = \sin^{2n}(w)\sum_{k\in Z}\frac{1}{(w+\pi k)^{2n}}$$

Differentiating the identity $\sum_{k\in Z}\dfrac{1}{w+k} = \cot w,\ (2n-1)$ *times,*

$$\sum_{k\in Z}\frac{1}{(w+\pi k)^{2n}} = \frac{-1}{(2n-1)!}\ \frac{d^{2n-1}}{dw^{2n-1}}(\cot w) \tag{1}$$

on substitution

$$\sum_{k\in Z}|\beta^n(2w+2\pi k)|^2 = \frac{-\sin^{2n}(w)}{(2n-1)!}\ \frac{d^{2n-1}}{dw^{2n-1}}(\cot w)$$

Put $n=1$ *in (1),*

$$\sum_{k\in Z}\frac{1}{(w+\pi k)^2} = \frac{1}{4}cosec^2(\frac{w}{2}) \tag{2}$$

Differentiating (2), $(2n-1)$ *times*

$$\sum_{k\in Z}\frac{1}{(w+\pi k)^{2n}} = \frac{1}{4(2n-1)!}\ \frac{d^{2n-2}}{dw^{2n-2}}(cosec^2(\frac{w}{2}))$$

For $n=2$, $\displaystyle\sum_{k\in Z}\frac{1}{(w+\pi k)^4} = \frac{1}{(2\sin\frac{w}{2})^4}(1-\frac{2}{3}\sin^2\frac{w}{2})$

$$\sum_{k\in Z}|\hat{\beta}^2(w+2\pi k)|^2 = 16\sin^4(\frac{w}{2})\sum_{k\in Z}\frac{1}{(w+2\pi k)^4} = 1-\frac{2}{3}\sin^2(\frac{w}{2}) \tag{3}$$

Theorem 3. *The wavelet* ψ *associated with the scaling function* φ *in* $L^2(\mathbb{R})$ *satisfy*

$$\hat{\psi}(w) = e^{i\frac{w}{2}}\frac{\sin^4(\frac{1}{4}w)}{(\frac{1}{4}w)^2}\left(\frac{1-\frac{2}{3}\cos^2(\frac{1}{4}w)}{(1-\frac{2}{3}\sin^2(\frac{1}{2}w))(1-\frac{2}{3}\sin^2(\frac{1}{4}w))}\right)^{\frac{1}{2}}$$

Proof: From (3) above

$$\sum_{k\in Z}|(1-\frac{2}{3}\sin^2(\frac{w}{2}))^{-\frac{1}{2}}\hat{\beta}_2(w+2\pi k)|^2 = 1$$

If we write $\hat{\phi} = (1-\frac{2}{3}\sin^2\frac{w}{2})^{-\frac{1}{2}}\hat{\beta}_2(w)$ then $\hat{\phi} = \left(\dfrac{\sin\frac{w}{2}}{\frac{w}{2}}\right)^2 m_{\phi}(w)$ where

$$m_{\phi}(w) = (1-\frac{2}{3}\sin^2\frac{w}{2})^{-\frac{1}{2}} \tag{4}$$

ISSN 2250-3153

The two scale equation for the scaling funtion ϕ with low pass filter $m_0(w)$ is $\hat{\phi}(2w) = m_0(w)\hat{\phi}(w)$. Using (4)

$$m_0(w) = \frac{(\sin w)^2 (1 - \frac{2}{3}\sin^2(\frac{1}{2}w))^{\frac{1}{2}}}{(2\sin(\frac{1}{2}w))^2 (1 - \frac{2}{3}\sin^2 w)^{\frac{1}{2}}}$$

$$= (\cos(\frac{w}{2}))^2 \left(\frac{1 - \frac{2}{3}\sin^2(\frac{1}{2}w)}{1 - \frac{2}{3}\sin^2 w} \right)^{\frac{1}{2}}$$

If the orthonormal wavelet associated with ϕ is ψ then ψ satisfies

$$\hat{\psi}(2w) = e^{iw}\overline{m_0(w+\pi)}\hat{\phi}(w)$$

Using above result we get

$$\hat{\psi}(2w) = e^{iw} \frac{\sin^4(\frac{1}{2}w)}{(\frac{1}{2}w)^2} \left(\frac{1 - \frac{2}{3}\cos^2(\frac{1}{2}w)}{(1 - \frac{2}{3}\sin^2 w)(1 - \frac{2}{3}\sin^2(\frac{1}{2}w))} \right)^{\frac{1}{2}} \quad (5)$$

The wavelet ψ is called *Franklin Wavelet*.

Remark: Now we turn on to F -spline and derive the *Franklin Fresnelet*, the wavelet associated with the F -spline. If ϕ is the scaling function associated with the F -spline $\hat{\beta}_\tau^2$ then (4) will become

$$\hat{\phi}(w) = (1 - \frac{2}{3}\sin^2(\frac{w}{2}))^{-\frac{1}{2}} \hat{\beta}_\tau^2(w)$$

$$= \hat{\beta}_2(w)\hat{k}_\tau(w)(1 - \frac{2}{3}\sin^2(\frac{w}{2}))^{-\frac{1}{2}} \quad \Theta \quad \hat{\beta}_\tau^2(w) = (\beta^2 * k_\tau)(w)$$

$$= e^{\frac{i\pi}{4}} e^{-i\pi(\tau w)^2} \left(\frac{\sin(\frac{w}{2})}{\frac{w}{2}} \right)^2 (1 - \frac{2}{3}\sin^2(\frac{w}{2}))^{-\frac{1}{2}}$$

Computing as in the previous case the $m_0(w)$ associated with this scaling function ϕ differ from the earlier one (4) by a factor of $e^{-i3\pi(\tau w)^2}$. Hence the Franklin Fresnlet ψ in the Fourier domain is expressed as in the RHS of (4) except for a multiple of $e^{-i3\pi(\tau w)^2}$.

ACKNOWLEDGMENT

The author received support in the form of a Minor Research Project from University Grants Commission, New Delhi, INDIA.

REFERENCES

[1] I.Daubechies, The Lectures on Wavelets,SIAM,Philadelphia,1992

[2] M. Liebling, Fresnelets, New Multi resolution Wavelet Bases for Digital Holography, IEEE Transactions on image processing, 2003

[3] David F Walnut, An Introduction to Wavelet Analysis, Birkhauser 2012

[4] G.Weiss and E.Hernandez,A First Course on Wavelets,CRC Press,1996

AUTHOR

M P Sivaramakrishnan, M.Sc.;Ph.D, Associate Professor in Mathematics,Government Brennen College,Thalassery,Kannur(District),Kerala(State),INDIA -PIN:670106.