ARC LENGTH of an ELLIPTICAL CURVE

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Abstract: In this paper, I have introduced a new patent rule for computing ARC LENGTH of an ELLIPTICAL CURVE. It is based on Geometrical Theorems. The method is fast and simplest of all other methods meant for Elliptical Arc Length. The earlier methods existing for computing Elliptical Arc Length like Riemann sum (by integration), Numerical Integration, Bernoulli's method, Euler's method, and other methods in this sequence till now, recently by Arvind Narayan(September-2012) provide approximate value and have more variables and involve more steps to compute.. The peculiarity of this method is that no smoothness (differentiability) of the curve is required, just the extremities of arc is enough to determine the exact arc length of the elliptical curve. The present method not only provides a formula, but also, it will serve as the precious tool for the subjects relevant to the Elliptical Arc Length.

Index Terms: patent rule, exact, geometrical theorems, peculiarity, approximate

I. Introduction

Earlier attempts to compute arc length of ellipse by antiderivative give rise to elliptical integrals (Riemann integrals) which is equally useful for calculating arc length of elliptical curves; though the latter is degree 3 or more, and the former is a degree 2 curves. Perhaps elliptical integrals are valuable tool, but for some curves it is difficult to evaluate and for some elliptical curves evaluation of elliptical integrals becomes impossible. Other methods like Riemann sum, and numerical integration after a long process gives an approximation, recently in 2012, Arvind narayan used geometry and trigonometry to find approximate elliptical arc length which requires end points of arc as well as their parametric equations.

The method which I am submitting is a simple solution for above problems, which just need the extremities of the elliptical arc. This method also establishes a valuable relation between elliptical arc length and its corresponding intercepted chord. To justify the significance of this method, it is necessary to explain it in two stages: Derivation of the formula, verification and its comparison.

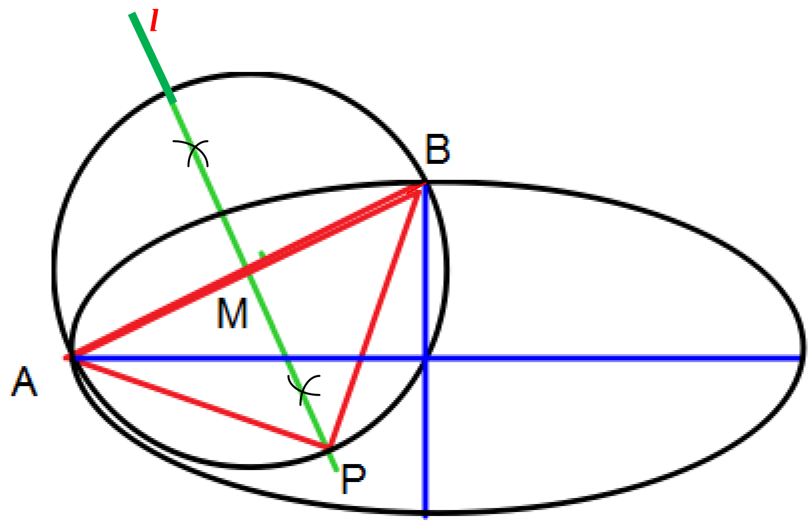
II. Derivation of the formula(Geometrically in the form of theorem)

[Farooque's Theorem]

Statement: The arc length($\boldsymbol{L}$) of an elliptical curve is equal to $\frac{\pi}{2\sqrt{2}}$ times of the intercepted chord length.

Given: An elliptical arc with extremities **A** (x_1, y_1) and **B** (x_2, y_2)

To Prove: $L = \frac{\pi}{2\sqrt{2}}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



Construction:

(i) Joining extremities A & B of given elliptical arc to make chord AB.

(ii) Draw perpendicular bisector (l) of the chord AB.

(iii) With mid- point M of chord AB, as center and AM as radius, draw a circle, which cuts the produced (l) at point P and Q.

(iv) Joining AP and BP.

Proof:

(A) In ΔAPM and ΔBPM,

PM=PM (Common side to both triangles)

AM=BM (Since M is the mid-point of AB)

$\angle$ AMP = $\angle$ BMP (Each is 90° (as PM $\perp$ AB))

By SAS Congruency,

ΔAPM≅ ΔBPM

∴ AP=BP (By cpct)

$\angle$ APM = $\angle$ BPM (By cpct)

Also $\angle$APB=90° [(As it is an angle in a semicircle (Appendix-A)]

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$$\therefore \angle APM = \frac{\angle APB}{2} = \frac{90^\circ}{2} = 45^\circ$$

(B) **Relation of chord AP and radius AM of the circle**

In ΔAPM, $\sin 45^\circ = \frac{AM}{AP}$

$$\frac{\mathbf{1}}{\sqrt{\mathbf{2}}} = \frac{AM}{AP}$$

$$\boldsymbol{AP = \sqrt{2}\, AM}$$

(C) Finding exact length of arc AB of Ellipse

Now consider sector APB

Since, Angle $= \frac{\text{Arc}}{\text{Radius}}$

$$\therefore\ \theta = \frac{L}{AP}$$

$$\frac{\pi}{2} = \frac{L}{\sqrt{2}\,AM}$$

$$L = \frac{\pi}{2} \times \sqrt{2} AM = \frac{\pi}{2} \times \sqrt{2} \times \frac{AB}{2} = \frac{\pi}{4} \times \sqrt{2} AB$$

$$L = \frac{\pi}{2\sqrt{2}} \times AB$$

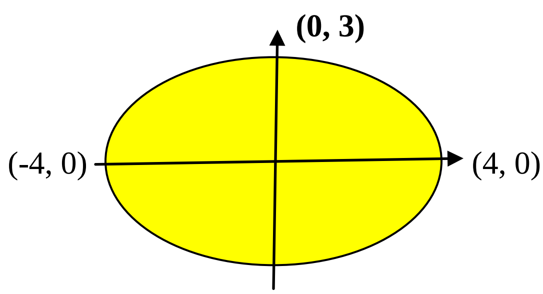
HERE, (Distance), AB=$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$\boldsymbol{L = \frac{\pi}{2\sqrt{2}} \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

NOTE:

The factor $\boldsymbol{\frac{\pi}{2\sqrt{2}} = 1.110720735 \approx 1.111}$ [which I must write F (farooque) factor], which when multiplied to the chord length between any two points on curve gives the length of corresponding arc.)

VERIFICATION: For example of an ellipse, $\frac{x^2}{16} + \frac{y^2}{9} = 1$



Circumference (C) of ellipse = $2\pi\sqrt{\frac{a^2+b^2}{2}}$

C= $2\pi\sqrt{\frac{4^2+3^2}{2}}$ = 22.21

By our recent formula,

AB= 5.55, which is the quarter of Circumference.

Now, C = 4×AB = 4×5.55 = 22.20

III. CONCLUSION

Therefore, above theorem gives a simple formula to determine the ‘‘**arc length of** given **Elliptical arc segment lying within a quadrant of the ellipse’’**, in the same manner, it is meant for other elliptical functions.

IV. PROSPECTS

Due to the simplicity and the degree of accuracy, this geometrical fact can be a step towards solving the problems related to elliptical Integrals, consequently a tool for all types of elliptic function. I hope that the method would find extensive applications where we require arc length of elliptical curves like cryptography and other applied engineering fields.

References:-

(1) Wikipedia Article: Ellipse.
(2) *Wikipedia Article: Elliptical Integral*
(3) *Wikipedia Article: Elliptical Curves*
(4) *Perimeter of Ellipse by Arvind Narayan (2012).*

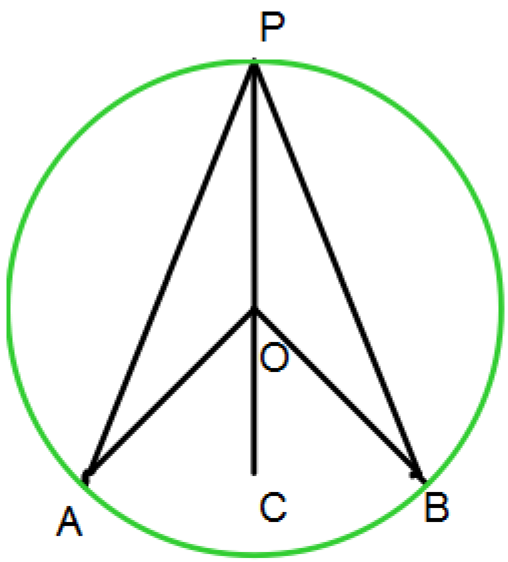
APPENDIX-A

STATEMENT: “Angle subtended at the centre of the circle by its arc is twice the angle which the same arc subtends at the remaining part of the circle.”

GIVEN: A circle with centre O and radius r, $\overset{\frown}{AB}$ is the arc and P is the point on the circle in its alternate segment.

TO PROVE: $\angle$AOB = 2 $\angle$APB

CONSTRUCTION: Draw a circle with centre O, AB is arc and joining OA, OB, also PO produced up to C and joining PA and PB.



PROOF: In $\triangle AOP$

$OA = OP$ (being radius of same circle)

$\angle OAP = \angle OPA$

We know that,

Exterior Angle = sum of two remote interior angles

$\angle AOC = \angle OAP + \angle OPA = 2\angle OPA$ ------- (1)

Similarly, in $\triangle BOP$

$\angle BOC = 2\angle BPO$ -------- (2)

Adding (1) and (2), we get

$\angle AOC + \angle BOC = 2(\angle APO + \angle BPO)$

$\angle AOB = 2\angle APB$ -------- (3)

<u>COROLLARY</u>: If AB is a semicircle then $\angle AOB = 180°$ ------- (4)

$2\angle APB = 180°$

$\angle APB = 90°$

Thus, angle in a semicircle is right angle.