On Hsu-Structure Manifold, Birecurrent and Symmetry

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Abstract- In this Paper, we have defined birecurrence and symmetry of different kinds in Hsu-structure manifold. Some theorem establishing relationship between different kinds of Birecurrent Hsu-structure manifold have been stated and proved Furthermore , Theorems on different kinds of Birecurrent and symmetric Hsu-structure manifold involving equivalent conditions with respect to various curvature tensors have been discussed. Birecurrence parameter have also been studied.

Index Terms- Birecurrence parameter, Curvature Tensors, C^{∞}-function, Hsu-structure manifold.

I. Introduction

If on an even dimensional manifold V_n, n = 2m of differentiability class C^{∞}, there exists a vector valued real linear function ϕ , satisfying

$$\phi^2 = a^r I_n , \quad (1.1)a$$

$$\overline{\overline{X}} = a^r X \text{ , for arbitrary vector field X.} \quad (1.1)b$$

where $\overline{X} = \phi X$, $0 \le r \le n$ and 'a' is a real or imaginary number.

Then {ϕ } is said to give to V_n a Hsu-structure defined by the equations (1.1) and the manifold V_n is called a Hsu-structure manifold.

*Remark (1.1):*The equation (1.1)a gives different structure for different values of 'a' and 'r'.

If $r = 0$, it is an almost product structure, if $a = 0$, it is an almost tangent structure, if $r = \pm 1$ and $a = +1$, it is an almost product structure, if $r = \pm 1$ and $a = -1$, it is an almost complex structure, if $r = 2$ then it is a GF-structure which includes π-structure for $a \neq 0$, an almost complex structure for $a = \pm i$,an almost product structure for $a = \pm 1$,an almost tangent structure for $a = 0$.

Let the Hsu-structure be endowed with a metric tensor g, such that

$$g(\overline{X}, \overline{Y}) + a^r g(X, Y) = 0 .$$

Then $\{\phi, g\}$ is said to give to V_n - metric Hsu-structure and V_n is called a metric Hsu-structure manifold.

Agreement(1.1): In what follows and the above , the equations containing X,Y,Z…….,etc.hold for these arbitrary vector in V_n.

The curvature tensor K, a vector -valued tri-linear function w.r.t. the connexion D is given by

$$K(X,Y,Z) = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z, \quad (1.2)a$$

where

$$[X,Y] = D_X Y - D_Y X. \quad (1.2)b$$

The Ricci tensor in V_n is given by

$$Ric(Y,Z) = (C_1^1 K)(Y,Z). \quad (1.3)$$

Where by $(C_1^1 K)(Y,Z)$, we mean the contraction of $K(X,Y,Z)$ with respect to first slot.
For Ricci tensor, we also have

$$Ric(Y,Z) = Ric(Z,Y), \tag{1.4)a}$$

$$Ric(Y,Z) = g(r(Y),Z) = g(Y,r(Z)), \tag{1.4)b}$$

$$(C_1^1 r) = R \tag{1.4)c}$$

Let W, C, L and V be the Projective, conformal, conharmonic and concircular curvature tensors respectively given by

$$W(X,Y,Z) = K(X,Y,Z) - \frac{1}{(n-1)}[Ric(Y,Z)X - Ric(X,Z)Y]. \tag{1.5}$$

$$C(X,Y,Z) = K(X,Y,Z) - \frac{1}{n-2}\{Ric(Y,Z)X - Ric(X,Z)Y - g(X,Z)r(Y) + g(Y,Z)r(X)\}$$
$$+ \frac{R}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]. \tag{1.6}$$

$$L(X,Y,Z) = K(X,Y,Z) - \frac{1}{(n-2)}[Ric(Y,Z)X - Ric(X,Z)Y - g(X,Z)r(Y) + g(Y,Z)r(X)]. \tag{1.7}$$

$$V(X,Y,Z) = K(X,Y,Z) - \frac{R}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{1.8}$$

A manifold is said to be recurrent, if

$$(\nabla K)(X,Y,Z,T) = A(T_1)K(X,Y,Z). \tag{1.9}$$

The recurrent manifold is said to be symmetric, if
$A(T_1) = 0,$ in the equation (1.9). (1.10)

A manifold is said to be birecurrent, if

$$(\nabla\nabla K)(X,Y,Z,T_1,T_2) = A(T_1,T_2)K(X,Y,Z). \tag{1.11}$$

It is said to be birecurrent symmetric, if

$$A(T_1,T_2) = 0. \tag{1.12}$$

II. BIRECURRENCE AND SYMMETRY OF DIFFERENT KINDS

Let P, a vector valued trilinear function, be any one of the curvature tensor K, W, C, L or V. Then we will define birecurrence in Q of different kinds as follows:
Definition (2.1): A Hsu-structure manifold is said to be (1)-birecurrent in P, if

$$a^r(\nabla\nabla P)(X,Y,Z,T_1,T_2) + (\nabla P)((\nabla\phi)(\overline{X},T_1),Y,Z,T_2) + (\nabla P)((\nabla\phi)(\overline{X},T_2),Y,Z,T_1) + P((\nabla\nabla\phi)(\overline{X},T_1,T_2),Y,Z)$$
$$= a^r A(T_1,T_2)P(X,Y,Z). \tag{2.1}$$

Where $A(T_1,T_2)$ is a non-vanishing C^∞ function, called birecurrence parameter.

Definition (2.2): A Hsu-structure manifold is said to be (12)-birecurrent in P, if

$$a^r(\nabla\nabla P)(X,\overline{Y},Z,T_1,T_2)+(\nabla P)((\nabla\phi)(\overline{X},T_1),\overline{Y},Z,T_2)+(\nabla P)(\nabla\phi)(\overline{X},T_2),\overline{Y},Z,T_1)$$
$$+a^r(\nabla P)(X,(\nabla\phi)(Y,T_1),Z,T_2)+a^r(\nabla P)(X,(\nabla\phi)(Y,T_2),Z,T_1)+P((\nabla\phi)(\overline{X},T_1),(\nabla\phi)(Y,T_2),Z)$$
$$+P((\nabla\phi)(\overline{X},T_2),(\nabla\phi)(Y,T_1),Z)+P((\nabla\nabla\phi)(\overline{X},T_1,T_2),\overline{Y},Z)+a^rP(X,(\nabla\nabla\phi)(Y,T_1,T_2),Z)$$
$$=a^rA(T_1,T_2)P(X,\overline{Y},Z). \tag{2.2)a$$

Or equivalently

$$a^r(\nabla\nabla P)(\overline{X},Y,Z,T_1,T_2)+a^r(\nabla P)((\nabla\phi)(X,T_1),Y,Z,T_2)+a^r(\nabla P)((\nabla\phi)(X,T_2),Y,Z,T_1)$$
$$+(\nabla P)(\overline{X},(\nabla\phi)(\overline{Y},T_1),Z,T_2)+(\nabla\phi)(\overline{X},(\nabla\phi)(\overline{Y},T_2),Z,T_1)+P((\nabla\phi)(X,T_1),(\nabla\phi)(\overline{Y},T_2),Z)$$
$$+P((\nabla\phi)(X,T_2),(\nabla\phi)(\overline{Y},T_1),Z)+a^rP((\nabla\nabla\phi)(X,T_1,T_2),Y,Z)+P(\overline{X},(\nabla\nabla\phi)(\overline{Y},T_1,T_2),Z)$$
$$=a^rA(T_1,T_2)P(\overline{X},Y,Z). \tag{2.2)b$$

Definition (2.3): A Hsu-structure manifold is said to be (123)-birecurrent in P, if

$$a^r(\nabla\nabla P)(X,\overline{Y},\overline{Z},T_1,T_2)+(\nabla P)((\nabla\phi)(\overline{X},T_1),\overline{Y},\overline{Z},T_2)+(\nabla P)((\nabla\phi)(\overline{X},T_2),\overline{Y},\overline{Z},T_1)$$
$$+a^r(\nabla P)(X,(\nabla\phi)(Y,T_1),\overline{Z},T_2)+a^r(\nabla P)(X,(\nabla\phi)(Y,T_2),\overline{Z},T_1)+a^r(\nabla P)(X,\overline{Y},(\nabla\phi)(Z,T_1),T_2)$$
$$+a^r(\nabla P)(X,\overline{Y},(\nabla\phi)(Z,T_2),T_1)+P(\nabla\phi)(\overline{X},T_1),(\nabla\phi)(Y,T_2),\overline{Z})+P((\nabla\phi)(\overline{X},T_2),(\nabla\phi)(Y,T_1),\overline{Z})$$
$$+P((\nabla\phi)(\overline{X},T_1),\overline{Y},(\nabla\phi)(Z,T_2))+P((\nabla\phi)(\overline{X},T_2),\overline{Y},(\nabla\phi)(Z,T_1))+a^rP(X,(\nabla\phi)(Y,T_1),(\nabla\phi)(Z,T_2))$$
$$+a^rP(X,(\nabla\phi)(Y,T_2),(\nabla\phi)(Z,T_1))+P((\nabla\nabla\phi)(\overline{X},T_1,T_2),\overline{Y},\overline{Z})+a^rP(X,(\nabla\nabla\phi)(Y,T_1,T_2),\overline{Z})$$
$$+a^rP(X,\overline{Y},(\nabla\nabla\phi)(Z,T_1,T_2))=a^rA(T_1,T_2)P(X,\overline{Y},\overline{Z}). \tag{2.3)a$$

Or equivalently

$$a^r(\nabla\nabla P)(\overline{X},Y,\overline{Z},T_1,T_2)+a^r(\nabla P)((\nabla\phi)(X,T_1),Y,\overline{Z},T_2)+a^r(\nabla P)((\nabla\phi)(X,T_2),Y,\overline{Z},T_1)$$
$$+(\nabla P)(\overline{X},(\nabla\phi)(\overline{Y},T_1),\overline{Z},T_2)+(\nabla P)(\overline{X},(\nabla\phi)(\overline{Y},T_2),\overline{Z},T_1)+a^r(\nabla P)(\overline{X},Y,(\nabla\phi)(Z,T_1),T_2)$$
$$+a^r(\nabla P)(\overline{X},Y,(\nabla\phi)(Z,T_2),T_1)+P(\nabla\phi)(X,T_1),(\nabla\phi)(\overline{Y},T_2),\overline{Z})+P((\nabla\phi)(X,T_2),(\nabla\phi)(\overline{Y},T_1),\overline{Z})$$
$$+a^rP((\nabla\phi)(X,T_1),Y,(\nabla\phi)(Z,T_2))+a^rP(\nabla\phi)(X,T_2),Y,(\nabla\phi)(Z,T_1))+P(\overline{X},(\nabla\phi)(\overline{Y},T_1),(\nabla\phi)(Z,T_2))$$
$$+P(\overline{X},(\nabla\phi)(\overline{Y},T_2),(\nabla\phi)(Z,T_1))+a^rP((\nabla\nabla\phi)(X,T_1,T_2),Y,\overline{Z})+a^rP(\overline{X},Y,(\nabla\nabla\phi)(Z,T_1,T_2))$$
$$+P(\overline{X},(\nabla\nabla\phi)(\overline{Y},T_1,T_2),\overline{Z})=a^rA(T_1,T_2)P(\overline{X},Y,\overline{Z}). \tag{2.3)b$$

Or equivalently

$$a^r(\nabla\nabla P)(\overline{X},\overline{Y},Z,T_1,T_2)+a^r(\nabla P)((\nabla\phi)(X,T_1),\overline{Y},Z,T_2)+a^r(\nabla P)((\nabla\phi)(X,T_2),\overline{Y},Z,T_1)$$
$$+a^r(\nabla P)(\overline{X},(\nabla\phi)(Y,T_1),Z,T_2)+a^r(\nabla P)(\overline{X},(\nabla\phi)(Y,T_2),Z,T_1)+(\nabla P)(\overline{X},\overline{Y},(\nabla\phi)(\overline{Z},T_1),T_2)$$
$$+(\nabla P)(\overline{X},\overline{Y},(\nabla\phi)(\overline{Z},T_2),T_1)+a^rP(\nabla\phi)(X,T_1),(\nabla\phi)(Y,T_2),Z)+a^rP((\nabla\phi)(X,T_2),(\nabla\phi)(Y,T_1),Z)$$
$$+P((\nabla\phi)(X,T_1),\overline{Y},(\nabla\phi)(\overline{Z},T_2))+P(\nabla\phi)(X,T_2),\overline{Y},(\nabla\phi)(\overline{Z},T_1))+P(\overline{X},(\nabla\phi)(Y,T_1),(\nabla\phi)(\overline{Z},T_2))$$
$$+P(\overline{X},(\nabla\phi)(Y,T_2),(\nabla\phi)(\overline{Z},T_1))+a^rP((\nabla\nabla\phi)(X,T_1,T_2),\overline{Y},Z)+P(\overline{X},\overline{Y},(\nabla\nabla\phi)(\overline{Z},T_1,T_2))$$
$$+a^rP(\overline{X},(\nabla\nabla\phi)(Y,T_1,T_2),Z)=a^rA(T_1,T_2)P(\overline{X},\overline{Y},Z). \tag{2.3)b$$

Note (2.1) Similarly (2) , (3) , (4) , (23) , (24) , (13) , (14) , (34) , (124) , (134) ,(234) birecurrence in P can also be defined.
Definition (2.4). A Hsu-structure manifold is said to be Ricci-(1)-birecurrent, if

$$a^r(\nabla\nabla Ric)(Y,Z,T_1,T_2)+(\nabla Ric)((\nabla\phi)(\overline{Y},T_1),Z,T_2)+(\nabla Ric)((\nabla\phi)(\overline{Y},T_2),Z,T_1)$$
$$+Ric((\nabla\nabla\phi)(\overline{Y},T_1,T_2)=a^r A(T_1,T_2)Ric(Y,Z) \quad (2.4)$$

Note (2.2) Similarly Ricci-(2)-birecurrent can also be defined.

Definition (2.5). A Hsu-structure manifold is said to Ricci-(12)-birecurrent if

$$a^r(\nabla\nabla Ric)(Y,\overline{Z},T_1,T_2)+(\nabla Ric)((\nabla\phi)(\overline{Y},T_1),\overline{Z},T_2)+(\nabla Ric)((\nabla\phi)(\overline{Y},T_2),\overline{Z},T_1)+a^r(\nabla Ric)(Y,(\nabla\phi)(Z,T_1),T_2)$$
$$+a^r(\nabla Ric)(Y,(\nabla\phi)(Z,T_2),T_1)+Ric((\nabla\phi)(\overline{Y},T_1),(\nabla\phi)(Z,T_2))+Ric((\nabla\phi)(\overline{Y},T_2),(\nabla\phi)(Z,T_1))$$
$$+Ric((\nabla\nabla\phi)(\overline{Y},T_1,T_2),\overline{Z})+a^r Ric(Y,(\nabla\nabla\phi)(Z,T_1,T_2))=a^r A(T_1,T_2)Ric(Y,\overline{Z}). \quad (2.5)a$$

Or equivalently

$$a^r(\nabla\nabla Ric)(\overline{Y},Z,T_1,T_2)+a^r(\nabla Ric)((\nabla\phi)(Y,T_1),Z,T_2)+a^r(\nabla Ric)((\nabla\phi)(Y,T_2),Z,T_1)$$
$$+(\nabla Ric)(\overline{Y},(\nabla\phi)(\overline{Z},T_1),T_2)+(\nabla Ric)(\overline{Y},(\nabla\phi)(\overline{Z},T_2),T_1)+Ric((\nabla\phi)(Y,T_1),(\nabla\phi)(\overline{Z},T_2))$$
$$+Ric((\nabla\phi)(Y,T_2),(\nabla\phi)(\overline{Z},T_1))+a^r Ric((\nabla\nabla\phi)(Y,T_1,T_2),Z)+Ric(\overline{Y},(\nabla\nabla\phi)(\overline{Z},T_1,T_2))=a^r A(T_1,T_2)Ric(\overline{Y},Z). \quad (2.5)b$$

Definition (2.6). A birecurrent Hsu-structure manifold is said to be P-birecurrence symmetric, if

$$A(T_1,T_2)=0. \quad (2.6)$$

Theorem (2.1): A P-(12)-birecurrent Hsu-structure manifold is P-(1)-birecurrent for the same recurrence parameter provided

$$a^r(\nabla P)(X,(\nabla\phi)(Y,T_1),Z,T_2)+a^r(\nabla P)(X,(\nabla\phi)(Y,T_2),Z,T_1)+P((\nabla\phi)(\overline{X},T_1),(\nabla\phi)(Y,T_2),Z)$$
$$+P((\nabla\phi)(\overline{X},T_2),(\nabla\phi)(Y,T_1),Z)+a^r P(X,(\nabla\nabla\phi)(Y,T_1,T_2),Z)=0. \quad (2.7)$$

Proof: Barring Y in equation (2.1), we get

$$a^r(\nabla\nabla P)(X,\overline{Y},Z,T_1,T_2)+(\nabla P)((\nabla\phi)(\overline{X},T_1),\overline{Y},Z,T_2)+(\nabla P)((\nabla\phi)(\overline{X},T_2),\overline{Y},Z,T_1)$$
$$+P((\nabla\nabla\phi)(\overline{X},T_1,T_2),\overline{Y},Z)=a^r A(T_1,T_2)P(X,\overline{Y},Z). \quad (2.8)$$

Using equation (2.7) in equation (2.2)a we get the equation (2.8), which shows that the P-(12)-birecurrent Hsu-structure manifold is P-(1)-birecurrent Hsu-structure manifold.

Theorem (2.2): A P-(123)-birecurrent Hsu-structure manifold is P-(12)-birecurrent for the same recurrence parameter provided

$$a^r(\nabla P)(X,\overline{Y},(\nabla\phi)(Z,T_1),T_2)+a^r(\nabla P)(X,\overline{Y},(\nabla\phi)(Z,T_2),T_1)+P((\nabla\phi)(\overline{X},T_1),\overline{Y},(\nabla\phi)(Z,T_2))$$
$$+P((\nabla\phi)(\overline{X},T_2),\overline{Y},(\nabla\phi)(Z,T_1))+a^r P(X,(\nabla\phi)(Y,T_1),(\nabla\phi)(Z,T_2))+a^r P(X,(\nabla\phi)(Y,T_2),(\nabla\phi)(Z,T_1))$$
$$+a^r P(X,\overline{Y},(\nabla\nabla\phi)(Z,T_1,T_2))=0. \quad (2.9)$$

Proof: Barring Z in equation (2.2)a , we get

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$$a^r(\nabla\nabla P)(X,\overline{Y},\overline{Z},T_1,T_2)+(\nabla P)((\nabla\phi)(X,T_1),\overline{Y},\overline{Z},T_2)+(\nabla P)(\nabla\phi)(\overline{X},T_2),\overline{Y},\overline{Z},T_1)$$
$$+a^r(\nabla P)(X,(\nabla\phi)(Y,T_1),\overline{Z},T_2)+a^r(\nabla P)(X,(\nabla\phi)(Y,T_2),\overline{Z},T_1)+P((\nabla\phi)(\overline{X},T_1),(\nabla\phi)(Y,T_2),\overline{Z})$$
$$+P((\nabla\phi)(\overline{X},T_2),(\nabla\phi)(Y,T_1),\overline{Z})+P((\nabla\nabla\phi)(\overline{X},T_1,T_2),\overline{Y},\overline{Z})+a^rP(X,(\nabla\nabla\phi)(Y,T_1,T_2),\overline{Z})$$
$$=a^rA(T_1,T_2)P(X,\overline{Y},\overline{Z}). \tag{2.10}$$

If the manifold is P-(123)-birecurrent then using equation (2.9) in equation (2.3)a, we get the equation (2.10), which shows that the P-(123)-birecurrent Hsu-structure manifold is P-(12)-birecurrent Hsu-structure manifold.

Theorem (2.3): A P-(123)-birecurrent Hsu-structure manifold is P-(1)-birecurrent for the same recurrence parameter, provided

$$a^r(\nabla P)(X,(\nabla\phi)(Y,T_1),\overline{Z},T_2)+a^r(\nabla P)(X,(\nabla\phi)(Y,T_2),\overline{Z},T_1)+a^r(\nabla P)(X,\overline{Y},(\nabla\phi)(Z,T_1),T_2)$$
$$+a^r(\nabla P)(X,\overline{Y},(\nabla\phi)(Z,T_2),T_1)+P((\nabla\phi)(\overline{X},T_1),(\nabla\phi)(Y,T_2),\overline{Z})+P((\nabla\phi)(\overline{X},T_2),(\nabla\phi)(Y,T_1),\overline{Z})$$
$$+P((\nabla\phi)(\overline{X},T_1),\overline{Y},(\nabla\phi)(Z,T_2)+P((\nabla\phi)(\overline{X},T_2),\overline{Y},(\nabla\phi)(Z,T_1)+a^rP(X,(\nabla\phi)(Y,T_1),(\nabla\phi)(Z,T_2))$$
$$+P(X,(\nabla\phi)(Y,T_2),(\nabla\phi)(Z,T_1))+a^rP(X,(\nabla\nabla\phi)(Y,T_1,T_2),\overline{Z})+a^rP(X,\overline{Y},(\nabla\nabla\phi)(Z,T_1,T_2)=0. \tag{2.11}$$

Proof: Barring Y and Z in equation (2.1), we get

$$a^r(\nabla\nabla P)(X,\overline{Y},\overline{Z},T_1,T_2)+(\nabla P)((\nabla\phi)(\overline{X},T_1),\overline{Y},\overline{Z},T_2)+(\nabla P)((\nabla\phi)(\overline{X},T_2),\overline{Y},\overline{Z},T_1)$$
$$+P((\nabla\nabla\phi)(\overline{X},T_1,T_2),\overline{Y},\overline{Z})=a^rA(T_1,T_2)P(X,\overline{Y},\overline{Z}). \tag{2.12}$$

If the manifold is (123)-birecurrent in P then using equation (2.11) in equation (2.3)a, we get the equation (2.12), which shows that the P-(123)-birecurrent Hsu-structure manifold is P-(1)-birecurrent Hsu-structure manifold.

Note (2.2): Theorems similar to theorems (2.1), (2.2) and (2.3) can also be stated and proved for (2) , (3) ,(13) , (23) -birecurrent Hsu-structure manifold.

Theorem (2.4): In a (1)-birecurrent Hsu-structure manifold , if any two of the following conditions hold for the same recurrence parameter then the third also hold:
(a) it is Conformal (1)-birecurrent,
(b) it is Conharmonic (1)-birecurrent,
(c) it is Concircular (1)-birecurrent.

Proof: From equation (1.6), (1.7) and (1.8) , we have

$$C(X,Y,Z)=L(X,Y,Z)+\frac{n}{n-2}\{K(X,Y,Z)-V(X,Y,Z)\}. \tag{2.13}$$

Barring X in equation (2.13), we get

$$C(\overline{X},Y,Z)=L(\overline{X},Y,Z)+\frac{n}{n-2}\{K(\overline{X},Y,Z)-V(\overline{X},Y,Z)\}. \tag{2.14}$$

Differentiating equation (2.14) with respect to T_1 and T_2 and then barring X in the resulting equation, we get

$$a^r(\nabla\nabla C)(X,Y,Z,T_1,T_2)+(\nabla C)((\nabla\phi)(\overline{X},T_1),Y,Z,T_2)+(\nabla C)((\nabla\phi)(\overline{X},T_2),Y,Z,T_1)$$
$$+C((\nabla\nabla\phi)(\overline{X},T_1,T_2),Y,Z)=a^r(\nabla\nabla L)(X,Y,Z,T_1,T_2)+(\nabla L)((\nabla\phi)(\overline{X},T_1),Y,Z,T_2)$$
$$+(\nabla L)((\nabla\phi)(\overline{X},T_2),Y,Z,T_1)+L((\nabla\nabla\phi)(\overline{X},T_1,T_2),Y,Z)+\frac{n}{(n-2)}\{a^r(\nabla\nabla K)(X,Y,Z,T_1,T_2)$$
$$+(\nabla K)((\nabla\phi)(\overline{X},T_1),Y,Z,T_2)+(\nabla K)((\nabla\phi)(\overline{X},T_2),Y,Z,T_1)+K((\nabla\nabla\phi)(\overline{X},T_1,T_2),Y,Z)$$
$$-a^r(\nabla\nabla\phi)(X,Y,Z,T_1,T_2)-(\nabla V)((\nabla\phi)(\overline{X},T_1),Y,Z,T_2)-(\nabla V)((\nabla\phi)(\overline{X},T_2),Y,Z,T_1)$$

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$$-V((\nabla\nabla\phi)(\overline{X},T_1,T_2),Y,Z)\}. \qquad (2.15)$$

Multiplying equation (2.13) by $a^r A(T_1,T_2)$ throughout subtracting the resulting equation from equation (2.15) and then using the fact the (1)-birecurrent Hsu-structure manifold is conformal (1)-birecurrent and conharmonic (1)-birecurrent, we have

$$a^r(\nabla\nabla V)(X,Y,Z,T_1,T_2)+(\nabla V)((\nabla\phi)(\overline{X},T_1),Y,Z,T_2)+(\nabla V)((\nabla\phi)(\overline{X},T_2),Y,Z,T_1)$$
$$+V((\nabla\nabla\phi)(\overline{X},T_1,T_2),Y,Z)=a^r A(T_1,T_2)V(X,Y,Z).$$

Which shows that the manifold is concircular (1)-birecurrent. The proof of the remaining two cases follows similarly.

Theorem (2.5): In a (1)-birecurrent symmetric Hsu-structure manifold if any two of the following conditions hold for the same recurrence parameter then the third also holds:

(a) it is Conharmonic (1)-birecurrent symmetric,
(b) it is Conformal (1)- birecurrent symmetric,
(c) it is Concircular (1)-birecurrent symmetric.

Proof: The statement follows from the theorem (2.4) and (1.12).

Theorem (2.6): In a (12)-birecurrent Hsu-structure manifold if any two of the following conditions hold for the same recurrence parameter then the third also holds:

(a) it is Conharmonic (12)-birecurrent ,
(b) it is Conformal (12)- birecurrent ,
(c) it is Concircular (12)-birecurrent.

Proof: Barring X and Y in equation (2.13), we get

$$C\left(\overline{X},\overline{Y},Z\right)=L\left(\overline{X},\overline{Y},Z\right)+\frac{n}{n-2}\left\{K\left(\overline{X},\overline{Y},Z\right)-V\left(\overline{X},\overline{Y},Z\right)\right\}. \qquad (2.16)$$

Differentiating equation (2.16) w.r.t T_1 .and T_2 successively using equation (2.16) and then barring X in the resulting equation, we get

$$a^r(\nabla\nabla C)(X,\overline{Y},Z,T_1,T_2)+(\nabla C)((\nabla\phi)(\overline{X},T_1),\overline{Y},Z,T_2)+(\nabla C)((\nabla\phi)(\overline{X},T_2),\overline{Y},Z,T_1)$$
$$+a^r(\nabla C)(X,(\nabla\phi)(Y,T_1),Z,T_2)+a^r(\nabla C)(X,(\nabla\phi)(Y,T_2),Z,T_1)+C((\nabla\phi)(\overline{X},T_1),(\nabla\phi)(Y,T_2),Z)$$
$$+C((\nabla\phi)(\overline{X},T_2),(\nabla\phi)(Y,T_1),Z)+C((\nabla\nabla\phi)(\overline{X},T_1,T_2),\overline{Y},Z)+a^r C(X,(\nabla\nabla\phi)(Y,T_1,T_2),Z)$$
$$=a^r(\nabla\nabla L)(X,\overline{Y},Z,T_1,T_2)+(\nabla L)((\nabla\phi)(\overline{X},T_1),\overline{Y},Z,T_2)+(\nabla L)((\nabla\phi)(\overline{X},T_2),\overline{Y},Z,T_1)$$
$$+a^r(\nabla L)(X,(\nabla\phi)(Y,T_1),Z,T_2)+a^r(\nabla L)(X,(\nabla\phi)(Y,T_2),Z,T_1)+L((\nabla\phi)(\overline{X},T_1),(\nabla\phi)(Y,T_2),Z)$$
$$+L((\nabla\phi)(\overline{X},T_2),(\nabla\phi)(Y,T_1),Z)+L((\nabla\nabla\phi)(\overline{X},T_1,T_2),\overline{Y},Z)+a^r L(X,(\nabla\nabla\phi)(Y,T_1,T_2),Z)$$
$$+\frac{n}{(n-2)}\{a^r(\nabla\nabla K)(X,\overline{Y},Z,T_1,T_2)+(\nabla K)((\nabla\phi)(\overline{X},T_1),\overline{Y},Z,T_2)+(\nabla K)((\nabla\phi)(\overline{X},T_2),\overline{Y},Z,T_1)$$
$$+a^r(\nabla K)(X,(\nabla\phi)(Y,T_1),Z,T_2)+a^r(\nabla K)(\overline{X},(\nabla\phi)(Y,T_2),Z,T_1)+K((\nabla\phi)(\overline{X},T_1),(\nabla\phi)(Y,T_2),Z)$$
$$+K((\nabla\phi)(\overline{X},T_2),(\nabla\phi)(Y,T_1),Z)+K((\nabla\nabla\phi)(\overline{X},T_1,T_2),\overline{Y},Z)+a^r K(X,(\nabla\nabla\phi)(Y,T_1,T_2),Z)$$
$$=a^r(\nabla\nabla V)(X,\overline{Y},Z,T_1,T_2)-(\nabla V)((\nabla\phi)(\overline{X},T_1),\overline{Y},Z,T_2)-(\nabla V)((\nabla\phi)(\overline{X},T_2),\overline{Y},Z,T_1)$$
$$-a^r(\nabla V)(X,(\nabla\phi)(Y,T_1),Z,T_2)-a^r(\nabla V)(X,(\nabla\phi)(Y,T_2),Z,T_1)-V((\nabla\phi)(\overline{X},T_1),(\nabla\phi)(Y,T_2),Z)$$
$$-V((\nabla\phi)(\overline{X},T_2),(\nabla\phi)(Y,T_1),Z)-V((\nabla\nabla\phi)(\overline{X},T_1,T_2),\overline{Y},Z)-a^r V(X,(\nabla\nabla\phi)(Y,T_1,T_2),Z).$$
$$(2.17)$$

Barring Y and multiplying equation (2.13) by $a^r A(T_1,T_2)$ throughout subtracting the resulting equation from equation (2.17) and then using the fact that the (12)-birecurrent Hsu-structure manifold is Conformal (12)-birecurrent and Conharmonic (12)-birecurrent, we have

$$a^r(\nabla\nabla V)(X,\overline{Y},Z,T_1,T_2)+(\nabla V)((\nabla\phi)(\overline{X},T_1),\overline{Y},Z,T_2)+(\nabla V)((\nabla\phi)(\overline{X},T_2),\overline{Y},Z,T_1)$$
$$+a^r(\nabla V)(X,(\nabla\phi)(Y,T_1),Z,T_2)+a^r(\nabla V)(X,(\nabla\phi)(Y,T_2),Z,T_1)+V((\nabla\phi)(\overline{X},T_1),(\nabla\phi)(Y,T_2),Z)$$
$$+V((\nabla\phi)(\overline{X},T_2),(\nabla\phi)(Y,T_1),Z)+V((\nabla\nabla\phi)(\overline{X},T_1,T_2),\overline{Y},Z)+a^r V(X,(\nabla\nabla\phi)(Y,T_1,T_2),Z)$$
$$=a^r A(T_1,T_2)V(X,\overline{Y},Z).$$

Which shows that the manifold is concircular (12)-birecurrent. The proof of the remaining two cases follows similarly.

Theorem (2.7): In a (12)-birecurrent symmetric Hsu-structure manifold if any two of the following conditions hold for the same recurrence parameter then the third also holds:

(a) it is Conharmonic (12)-birecurrent symmetric,
(b) it is Conformal (12)- birecurrent symmetric,
(c) it is Concircular (12)-birecurrent symmetric.

Proof: The statement follows from the theorem (2.6) and (1.12).

Note (2.3): Similarly type of theorems can also be stated and proved for (2) , (3) , (13) , (23) and (123) - birecurrent and birecurrent symmetric Hsu-structure manifold.

III. RECURRENCE TENSOR FIELD

Theorem (3.1): In birecurrent Hsu-structure manifold the recurrence tensor field $A(T_1,T_2)$ is non-symmetric.

Proof: Let the manifold is (1)-birecurrent then from equation (2.1) , we have

$$a^r(\nabla\nabla P)(X,Y,Z,T_1,T_2)+(\nabla P)((\nabla\phi)(\overline{X},T_1),Y,Z,T_2)+(\nabla P)((\nabla\phi)(\overline{X},T_2),Y,Z,T_1)$$
$$+P((\nabla\nabla\phi)(\overline{X},T_1,T_2),Y,Z)=a^r A(T_1,T_2)P(X,Y,Z). \tag{3.1}$$

Interchanging T_1 and T_2 in equation (3.1) then subtracting the resulting equation from equation (3.1), we get

$$a^r((\nabla\nabla K)(X,Y,Z,T_1,T_2)-(\nabla\nabla K)(X,Y,Z,T_1,T_2)+K((\nabla\nabla\phi)(\overline{X},T_1,T_2),Y,Z)$$
$$-((\nabla\nabla\phi)(\overline{X},T_1,T_2),Y,Z)-a^r((A(T_1,T_2)-A(T_2,T_1))K(X,Y,Z). \tag{3.2}$$

Using Ricci identities for K(X,Y,Z) in equation (3.2) and then using equation (1.1)a in the resulting equation , we get

$$a^r K(T_2,T_1,K(X,Y,Z))-a^r K(X,K(T_2,T_1,Y),Z)-a^r K(X,Y,K(T_2,T_1,Z))$$
$$-K(\overline{X(T_2,T_1,\overline{X})},Y,Z)=a^r((A(T_1,T_2)-A(T_2,T_1))K(X,Y,Z). \tag{3.3}$$

Similarly, if the manifold is (12) or (123) - birecurrent then, we have

$$a^r K(T_2,T_1,K(X,\overline{Y},Z))-a^r K(X,\overline{Y},K(T_2,T_1,Z))-K(\overline{K(T_2,T_1,\overline{X})},\overline{Y},Z)$$
$$-a^r K(X,\overline{K(T_2,T_1,Y)},Z)=a^r((A(T_1,T_2)-A(T_2,T_1))K(X,\overline{Y},Z). \tag{3.4}$$

Or

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$$a^r K(T_2,T_1,K(X,\overline{Y},\overline{Z})) - a^r K(X,\overline{Y},\overline{K(T_2,T_1,Z)}) - K(\overline{K(T_2,T_1,\overline{X})},\overline{Y,Z})$$
$$- a^r K(X,\overline{K(T_2,T_1,Y)},\overline{Z}) = a^r((A(T_1,T_2) - A(T_2,T_1))K(X,\overline{Y,Z}). \tag{3.5}$$

Similarly equations can be obtained for (2), (3), (23) and (13) - birecurrent Hsu-structure manifold.
The equations (3.3) , (3.4) , and (3.5) prove the statement.

Theorem (3.2): In a (1)-birecurrent Hsu-structure manifold, the recurrence tensor field $A(T_1,T_2)$ satisfies the relation

$$a^r A(U)(A(T_1,T_2) - A(T_2,T_1))K(X,Y,Z) - K((\nabla\phi)(K(T_2,T_1,\overline{X}),U),Y,Z)$$
$$- K((\overline{K(T_2,T_1,(\nabla\phi)(X,U)},Y,Z) = a^r((\nabla A)(T_1,T_2,U) - ((\nabla A)(T_2,T_1,U))K(X,Y,Z). \tag{3.6}$$

Proof: Differentiating equation (3.3) covariently and using equations (1.9) and (3.3) in the resulting equation, we get the equation (3.6).

Theorem (3.3): If the Hsu-structure manifold is birecurrent and (1)-birecurrent for the same recurrence parameter, then we have

$$a^r A(U)((A(T_1,T_2) - A(T_2,T_1)) = a^r((\nabla A)(T_1,T_2,U) - (\nabla A)(T_2,T_1,U)). \tag{3.7}$$

Proof: From equation (1.11), we have

$$(\nabla\nabla K)(X,Y,Z,T_1,T_2) = A(T_1,T_2)K(X,Y,Z). \tag{3.8}$$

Interchanging T_1 and T_2 in equation (3.8) and subtracting the resulting equation from equation (3.8), we get

$$(\nabla\nabla K)(X,Y,Z,T_1,T_2) - (\nabla\nabla K)(X,Y,Z,T_2,T_1) = (A(T_1,T_2) - A(T_2,T_1))K(X,Y,Z). \tag{3.9}$$

Using Ricci identities in equation (3.9) and then comparing the resulting equation with the equation (3.3), we get

$$K(\overline{K(T_2,T_1,\overline{\overline{X}}}),Y,Z) = a^r K(K(T_2,T_1,X),Y,Z). \tag{3.10}$$

Using equation (3.10) in equation (3.3), we get

$$a^r K(T_2,T_1,K(X,Y,Z)) - a^r K(X,K(T_2,T_1,Y),Z) - a^r K(X,Y,K(T_2,T_1,Z))$$
$$- a^r K(T_2,T_1,X),Y,Z) - a^r((A(T_1,T_2) - A(T_2,T_1))K(X,Y,Z). \tag{3.11}$$

Differentiating equation (3.11) covariently and using equation (1.9) and (3.11), we have

$$a^r A(U)((A(T_1,T_2) - A(T_2,T_1)) = a^r((\nabla A)(T_1,T_2,U) - (\nabla A)(T_2,T_1,U)).$$

Hence we have the statement.

Theorem (3.4): If the Hsu-structure manifold is birecurrent and (1)-birecurrent for the same recurrence parameter, then

$$a^r(A(K)U,J,T_2),T_1) + A(T_2,K(U,J,T_1) - A(K(U,J,T_1),T_2) - A(T_1,K(U,J,T_2)$$
$$= a^r(A(T_1,T_2) - A(T_2,T_1))((\nabla A)(U,J) - (\nabla A)(J,U)). \tag{3.12}$$

Proof: Differentiating equation (3.7) covariently and using equation (3.7), we get

$$a^r(\nabla\nabla A)(T_1,T_2,U,J)-a^r(\nabla\nabla A)(T_2,T_1,U,J)=a^r(A(T_1,T_2)-A(T_2,T_1))(\nabla A)(U,J)$$
$$+a^r(A(T_1,T_2)-A(T_2,T_1))A(U,J). \tag{3.13}$$

Interchanging U and J in equation (3.13) and then subtracting the resulting equation obtained from equation (3.13), we get

$$a^r(\nabla\nabla A)(T_1,T_2,U,J)-a^r(\nabla\nabla A)(T_2,T_1,U,J)=a^r(A(T_1,T_2)-A(T_2,T_1))(\nabla A)(U,J)$$
$$-a^r(\nabla A)(J,U)+a^r(\nabla\nabla A)(T_1,T_2,J,U)-a^r(\nabla\nabla A)(T_1,T_2,J,U). \tag{3.14}$$

Also from Ricci-identities for $A(T_1,T_2)$, we have

$$(\nabla\nabla A)(T_1,T_2,U,J)-(\nabla\nabla A)(T_2,T_1,J,U)=-A(K(U,J,T),T_2)-A(T_2,K(U,J,T_2)). \tag{3.15}$$

Interchanging T_1 and T_2 in equation (3.15) and then subtracting the resulting equation obtained from equation (3.15), we get

$$(\nabla\nabla A)(T_1,T_2,U,J)-(\nabla\nabla A)(T_2,T_1,U,J)=A(K(U,J,T_1,T_2)-A(T_1,K(U,J,T_2)$$
$$+A(K(U,J,T_2),T_1)+A(T_2,K(U,J,T_1)+(\nabla\nabla A)(T_1,T_2,J,U)-(\nabla\nabla A)(T_2,T_1,J,U).$$

From equations (3.14) and (3.16), we get the equation (3.12).
Note (3.1): Theorems of the type (3.2), (3.3) and (3.4) can also be proved taking (2) or (3) Birecurrent Hsu-structure manifold instead of (1)-birecurrent Hsu-structure manifold.

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