Coupled fixed point theorems for generalized 
\((\alpha, \psi)\)-contractive type maps

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Abstract: In this paper, we introduce generalized \((\alpha, \psi)\)-contractive maps and prove the existence and uniqueness of coupled fixed points for generalized \((\alpha, \psi)\)-contractive maps in \(G\)-metric spaces. Furthermore, we provide examples in support of our results.

Keywords: \(\alpha\)-admissible, \((\alpha, \psi)\)-contractive maps, mixed monotone property, coupled fixed point, generalized \((\alpha, \psi)\)-contractive map.


1 Introduction

Fixed points and fixed point theorems have always a prominent role to find the existence of solutions of problems that arise in theoretical mathematics. In 1922, Banach [2] proved a remarkable results in this direction that each contraction in a complete metric space has a unique fixed point. Later many authors have directed their attention to this concept and have generalized the Banach fixed point theorems in various ways. In 2012, Samet, Vetro and Vetro [7] introduced a new concept namely \((\alpha, \psi)\)-contractive mappings and proved the related fixed points of such mappings in metric space setting.

Recently, Mustafa and Sims [4] introduction a new concept namely generalized metric space called \(G\)-metric space and characterized Banach fixed point theorem in the context of \(G\)-metric space. For more works on the existence of fixed points and coupled fixed points in \(G\)-metric spaces, we refer [4].

In 1987, Guo and Lakshmi Kantham [12] introduced the notion of a coupled fixed points for mixed monotone operators. The concept of a coupled fixed point was reconsidered by Gnana-Bhaskar and Lakshmi Kantham [11] in 2006. They proved...
and discussed the existence and uniqueness of a coupled fixed point of an operator $F : X \times X \to X$ on a partially ordered metric spaces, we refer [11].

Later, Alghamdi and Karapinar [5] introduced the new concept namely $(G, \beta, \psi)$-contractive type maps which are generalizations of $(\alpha, \psi)$-contractive maps, proved existence and uniqueness of fixed points of such contractive maps in $G$-metric spaces.

## 2 Preliminaries

Throughout this paper we denote by $\Psi$ the family of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ which satisfies $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$ where $\psi^n$ is the $n^{th}$ iterate of $\psi$.

**Remark 2.1.** Any function $\psi \in \Psi$ satisfies $\lim_{n\to\infty} \psi^n(t) = 0$, $\psi(t) < t$ for any $t > 0$ and $\psi$ is continuous at 0.

**Definition 2.2.** (Samet, Vetro and Vetro [7, Definition 2.1]) Let $(X, d)$ be a metric space and $T : X \to X$. We say that $T$ is $(\alpha, \psi)$-contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that $\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$.

$$\psi$$

**Definition 2.3.** (Samet, Vetro and Vetro [7, Definition 2.1]) Let $(X, d)$ be a metric space, $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. We say that $T$ is $\alpha$-admissible if $x, y \in X$ $\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$.

For examples on $\alpha$-admissible functions, we refer [7] and for more works on $\alpha$-admissible functions, we refer [8], [10], [9], [6].

**Theorem 2.4.** (Samet, Vetro and Vetro [7, Theorem 2.1]) Let $(X, d)$ be a complete metric space and $T : X \to X$. Suppose that there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that $T$ is $(\alpha, \psi)$-contractive map.

Also, assume that

(i) $T$ is $\alpha$-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$; either

(iii) $T$ is continuous; (or)

(iv) if $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n$.

Then $T$ has a fixed point. i.e., there exists $u \in X$ such that $Tu = u$.

**Definition 2.5.** (Karapinar and Samet [8, Definition 2.1]) Let $(X, d)$ be a metric space and $T : X \to X$ be a given mapping. We say that $T$ is a generalized $(\alpha, \psi)$-contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that for all $x, y \in X$

$$\alpha(x, y)d(Tx, Ty) \leq \psi(M(x, y)),$$

where

$$M(x, y) = \max\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\}. 
$$

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**Theorem 2.6.** (Karapinar and Samet [8, Theorem 2.3]) Let \((X, d)\) be a complete metric space and \(T : X \to X\). Suppose that there exist two functions \(\alpha : X \times X \to [0, \infty)\) and \(\psi \in \Psi\) such that \(T\) is a generalized \((\alpha, \psi)\)-contractive map. Also, assume that the following conditions are satisfied:

(i) \(T\) is \(\alpha\)-admissible;

(ii) there exists \(x_0 \in X\) such that \(\alpha(x_0, Tx_0) \geq 1\); either

(iii) \(T\) is continuous; (or)

(iv) if \(\{x_n\}\) is a sequence in \(X\) such that \(\alpha(x_n, x_{n+1}) \geq 1\) for all \(n\) and \(x_n \to x\) as \(n \to \infty\), then there exists a subsequence \(\{x_{n_k}\}\) of \(\{x_n\}\) such that \(\alpha(x_{n_k}, x) \geq 1\) for all \(k\).

Then there exists \(u \in X\) such that \(Tu = u\).

Mustafa and Sims [4] introduced the concept of G-metric space and proved fixed point results in complete G-metric spaces. After that, Alghamdi and Karapinar [5] proved some fixed point results in complete G-metric spaces.

**Definition 2.7.** [11] Let \((X, \preceq)\) be a partially ordered set and \(F : X \times X \to X\) the mapping \(F\) is said to have the mixed monotone property if \(F(x, y)\) is monotone non-decreasing in \(x\) and monotone non-increasing in \(y\), that is for any \(x, y \in X\), \(x_1, x_2 \in X\), \(x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y)\)

\(y_1, y_2 \in X\), \(y_1 \preceq y_2 \Rightarrow F(x, y_1) \preceq F(x, y_2)\).

**Definition 2.8.** [11] An element \((x, y) \in X \times X\) is called a coupled fixed point of the mapping \(F : X \times X \to X\) if \(F(x, y) = x, F(y, x) = y\).

**Definition 2.9.** [11] Let \(X\) be a non-empty set and \(F : X \times X \to X\) be a mapping. An element \(x \in X\) is called a fixed point of \(F\) if \(x = F(x, x)\).

**Theorem 2.10.** [11] Let \((X, \preceq)\) be a partially ordered set and suppose that there is a metric \(d\) on \(X\) such that \((X, d)\) is a complete metric space. Let \(F : X \times X \to X\) be a mapping having mixed monotone property on \(X\). Assume that there exists \(k \in [0, 1)\) such that

\(d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(x, u) + d(y, v)]\)

for all \(x, y, u, v \in X\) with \(x \succeq u\) and \(y \succeq v\). Suppose that either

(i) \(F\) is continuous (or);

(ii) \(X\) has the following property

(a) if \(\{x_n\}\) is a non-decreasing sequence with \(\{x_n\} \to x\) then \(\{x_n\} \leq x\) for all \(n\).

(b) if \(\{x_n\}\) is a non-decreasing sequence with \(\{x_n\} \to x\) then \(\{x_n\} \leq x\) for all \(n\).
Further, if there exist \( x_0, y_0 \in X \) such that \( x_0 \leq F(x_0, y_0) \) and \( y_0 \geq F(y_0, x_0) \), then there exist \( x, y \in X \) such that \( x = F(x, y) \) and \( y = F(y, x) \). i.e., \( F \) has a coupled fixed point in \( X \).

Lemma 2.11. \([6]\) Let \( F : X \times X \rightarrow X \) be a given mapping. Define the mapping \( T_F : X \times X \rightarrow X \times X \) by \( T_F(x, y) = (F(x, y), F(y, x)) \) for all \( (x, y) \in X \times X \). Then \( (x, y) \) is fixed point of \( T_F \) if and only if \( (x, y) \) is a coupled fixed point of \( F \).

Lemma 2.12. \([6]\) Let \( (X, G) \) be a \( G \)-metric space. A mapping \( F : X \times X \rightarrow X \) is said to be continuous if for any two \( G \)-Convergent sequence \( \{x_n\} \) and \( \{y_n\} \) converging to \( x \) and \( y \), respectively, \( \{F(x_n, y_n)\} \) is \( G \)-converging to \( F(x, y) \).

Alghamdi and Karapinar \([5]\) proved the following results.

Theorem 2.13. Let \( (X, G) \) be a complete \( G \)-metric space and let \( F : X \times X \rightarrow X \) be a given mapping. Suppose there exist \( \psi \in \Psi \) and the function \( \beta : X^2 \times X^2 \rightarrow [0, \infty) \) such that
\[
\beta((x, y), (u, v), (u, v))G(F(x, y), F(u, v), F(u, v)) \leq \frac{1}{2}(\psi(G(\beta(x, u, v)+G(y, v, v)))) \tag{2.13.1}
\]
for all \( (x, y), (u, v) \in X \times X \). Suppose that

(i) for all \( (x, y), (u, v) \in X \times X \), we have
\[
\beta((x, y), (u, v), (u, v)) \geq 1 \text{ implies } \\
\beta(F(x, y), F(y, x), (F(u, v), F(v, u)), (F(u, v), F(v, u))) \geq 1
\]
(ii) there exist \( (x_0, y_0) \in X \times X \) such that
\[
\beta((x_0, y_0), (F(x_0, y_0), F(y_0, x_0)), (F(x_0, y_0), (F(y_0, x_0))) \geq 1 \text{ and } \\
\beta((F(y_0, x_0), (F(x_0, y_0), (F(y_0, x_0), (F(x_0, y_0), (F(x_0, y_0)))) \geq 1
\]
(iii) \( F \) is continuous; or

(iv) if \( \{x_n\} \) and \( \{y_n\} \) are sequences in \( X \) such that
\[
\beta((x_n, y_n), (x_{n+1}, y_{n+1}), (x_{n+1}, y_{n+1})
\]
then \( F \) has a coupled fixed point. i.e., there exist \( (x^*, y^*) \in X \times X \) such that
\[
F(x^*, y^*) = x^* \text{ and } F(y^*, x^*) = y^*.
\]

Definition 2.14. \([12]\) Let \( X \) be a nonempty set \( F : X \times X \rightarrow X \) and \( g : X \rightarrow X \) then

(i) An element \( (x, y) \in X \times X \) is called a coupled coincidence point of the mappings \( F \) and \( g \) if \( F(x, y) = g(x) \) and \( F(y, x) = g(y) \).

(ii) An element \( (x, y) \in X \times X \) is called a common coupled coincidence point of the mappings \( F \) and \( g \) if \( F(x, y) = g(x) = x \) and \( F(y, x) = g(y) = y \).
3 Main Result

We introduce the concept of $F$ is generalized $(\alpha, \psi)$-contractive type mappings as follows:

**Definition 3.1.** Let $(X, G)$ be a $G$-metric space. Let $F : X \times X \to X$ be a map if there exist two functions $\alpha : X^2 \times X^2 \times X^2 \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\begin{cases}
\alpha((x, y), (u, v))(u, v)G(x, y, F(x, y), F(u, v), F(0, 0)) \leq \psi(G(x, u, u) + G(y, v, v), \frac{1}{2}(G(x, F(x, y), F(0, 0))) + G(y, F(y, x), F(0, 0))) + (G(u, F(u, v), F(0, 0)) + G(v, F(v, u), F(0, 0))) + G(y, F(v, u), F(v, v)) + (G(u, F(x, y), F(x, y)) + G(v, F(y, x), F(y, x)))
\end{cases}$$

(3.1.1)

for all $(x, y), (u, v) \in X \times X$.

then we say that $F$ is generalized $(\alpha, \psi)$-contractive map in two variables.

**Example 3.2.** Let $X = \{0, 1, 3\}$. We define $G : X^3 \to \mathbb{R}_+$ by

$G(x, y, z) = d(x, y) + d(y, z) + d(z, x)$. Let $A = \{(3, 0), (0, 3)\}, B = \{(0, 0), (1, 1), (3, 3), (3, 1), (1, 3)\}$

and $C = \{(1, 0), (0, 1)\}$. We define mapping $F : X \to X$ by $F(x, y) = \begin{cases}
1 & \text{if } (x, y) \in A \\
0 & \text{if } (x, y) \in B \\
3 & \text{if } (x, y) \in C.
\end{cases}$

We define $\alpha : X^3 \to [0, \infty)$ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\alpha((x, y), (u, v), (u, v)) = \begin{cases}
\frac{6}{5} & \text{if } (x, y) \in A, (u, v) \in B \\
0 & \text{otherwise}
\end{cases} \quad \text{and } \psi(t) = \frac{1 + t}{1 + t^2} \text{ for all } t > 0.$$

Now, we verify the inequality (3.1.1) as follows:

**Case (i):** $(x, y) = (3, 0)$ and $(u, v) = (0, 0)$

In this case, $F(3, 0) = F(0, 3) = 1, F(0, 0) = 0, \alpha(3, 0), (0, 0), (0, 0)) = \frac{6}{5}$ and

$M((3, 0), (0, 0)) = 6$

$$\alpha((x, y), (u, v), (u, v))G((x, y), (u, v), (u, v)) = \alpha((3, 0), (0, 0), (0, 0))G((3, 0), (0, 0), (0, 0)) = \frac{6}{5} \leq \frac{1}{2}(\psi(M((3, 0), (0, 0)))) = \frac{1}{2}(\psi(M((x, y), (u, v))))).$$

**Case (ii):** $(x, y) = (3, 0)$ and $(u, v) = (1, 1)$

In this case, $F(3, 0) = F(0, 3) = 1, F(1, 1) = 0, \alpha((3, 0), (1, 1), (1, 1)) = \frac{6}{5}$ and

$M((3, 0), (1, 1)) = 6$

$$\alpha((x, y), (u, v), (u, v))G((x, y), (u, v), (u, v)) = \alpha((3, 0), (1, 1), (1, 1))G((3, 0), (1, 1), (1, 1)) = \frac{12}{5} \leq \frac{1}{2}(\psi(M((3, 0), (1, 1)))) = \frac{1}{2}(\psi(M((x, y), (u, v))))).$$

**Case (iii):** $(x, y) = (3, 0)$ and $(u, v) = (3, 3)$

In this case, $F(3, 0) = F(0, 3) = 1, F(3, 3) = 0, \alpha((3, 0), (3, 3), (3, 3)) = \frac{6}{5}$ and

$M((3, 0), (3, 3)) = 10$

$$\alpha((x, y), (u, v), (u, v))G((x, y), (u, v), (u, v)) = \alpha((3, 0), (3, 3), (3, 3))G((3, 0), (3, 3), (3, 3)) = \frac{12}{5} \leq \frac{1}{2}(\psi(M((3, 0), (3, 3)))) = \frac{1}{2}(\psi(M((x, y), (u, v))))).$$

**Case (iv):** $(x, y) = (3, 0)$ and $(u, v) = (3, 1)$

In this case, $F(3, 0) = F(0, 3) = 1, F(3, 1) = F(1, 3) = 0, \alpha((3, 0), (3, 1), (3, 1)) = \frac{6}{5}$ and

$M((3, 0), (3, 1)) = 6$

$$\alpha((x, y), (u, v), (u, v))G((x, y), (u, v), (u, v)) = \alpha((3, 0), (3, 1), (3, 1))G((3, 0), (3, 1), (3, 1)) = \frac{12}{5} \leq \frac{1}{2}(\psi(M((3, 0), (3, 1)))) = \frac{1}{2}(\psi(M((x, y), (u, v))))).$$

**Case (v):** $(x, y) = (3, 0)$ and $(u, v) = (1, 3)$

In this case, $F(3, 0) = F(0, 3) = 1, F(3, 1) = F(1, 3) = 0, \alpha((3, 0), (1, 3), (1, 3)) = \frac{6}{5}$

In this case, $F(3, 0) = F(0, 3) = 1, F(3, 1) = F(1, 3) = 0, \alpha((3, 0), (1, 3), (1, 3)) = \frac{6}{5}$
and $M((3,0),(1,3)) = 10$

$$\alpha((x,y),(u,v),(u,v)) = \alpha((3,0),(1,3)) = 1$$

$$\frac{15}{7} \leq \frac{1}{2}(\psi(M((3,0),(1,3)))) = \frac{1}{2}(\psi(M((x,y),(u,v))))$$.

**Case (vi):** $(x,y) = (0,3)$ and $(u,v) = (0,0)$

In this case, $F(0,3) = F(3,0) = 1$, $F(0,0) = 0$, $\alpha((0,3),(0,0),0)) = \frac{6}{7}$ and $M((0,3),(0,0)) = 6$

$$\alpha((x,y),(u,v),(u,v)) = \alpha((0,3),(0,0),0)) = \frac{15}{7} \leq \frac{1}{2}(\psi(M((0,3),(0,0)))) = \frac{1}{2}(\psi(M((x,y),(u,v))))$$.

**Case (vii):** $(x,y) = (0,3)$ and $(u,v) = (1,1)$

In this case, $F(0,3) = F(3,0) = 1$, $F(1,1) = 0$, $\alpha((0,3),(1,1),(1,1)) = \frac{6}{7}$ and $M((0,3),(1,1)) = 6$

$$\alpha((x,y),(u,v),(u,v)) = \alpha((0,3),(1,1),(1,1)) = \frac{15}{7} \leq \frac{1}{2}(\psi(M((0,3),(1,1)))) = \frac{1}{2}(\psi(M((x,y),(u,v))))$$.

**Case (viii):** $(x,y) = (0,3)$ and $(u,v) = (3,3)$

In this case, $F(0,3) = F(3,0) = 1$, $F(3,3) = 0$, $\alpha((0,3),(3,3),(3,3)) = \frac{6}{7}$ and $M((0,3),(3,3)) = 9$

$$\alpha((x,y),(u,v),(u,v)) = \alpha((0,3),(3,3),(3,3)) = \frac{15}{7} \leq \frac{1}{2}(\psi(M((0,3),(3,3)))) = \frac{1}{2}(\psi(M((x,y),(u,v))))$$.

**Case (ix):** $(x,y) = (0,3)$ and $(u,v) = (3,1)$

In this case, $F(3,0) = F(0,3) = 1$, $F(3,1) = F(1,3) = 0$, $\alpha((0,3),(3,1),(3,1)) = \frac{6}{7}$ and $M((0,3),(3,1)) = 10$

$$\alpha((x,y),(u,v),(u,v)) = \alpha((0,3),(3,1),(3,1)) = \frac{15}{7} \leq \frac{1}{2}(\psi(M((0,3),(3,1)))) = \frac{1}{2}(\psi(M((x,y),(u,v))))$$.

**Case (x):** $(x,y) = (0,3)$ and $(u,v) = (1,3)$

In this case, $F(3,0) = F(0,3) = 1$, $F(3,1) = F(1,3) = 0$, $\alpha((0,3),(1,3),(1,3)) = \frac{6}{7}$ and $M((0,3),(1,3)) = 10$

$$\alpha((x,y),(u,v),(u,v)) = \alpha((0,3),(1,3),(1,3)) = \frac{15}{7} \leq \frac{1}{2}(\psi(M((0,3),(1,3)))) = \frac{1}{2}(\psi(M((x,y),(u,v))))$$.

Therefore, the inequality (3.1.1) satisfies. Hence $F$ is generalized $(\alpha, \psi)$-contractive map in two variables.

Here, we observe that the inequality (2.13.1) fails to hold. For, by choosing $(x,y) = (3,0)$ and $(u,v) = (3,1)$ we have $\alpha((3,0),(3,1),(3,1))G((3,0),(3,1),(3,1)) = \frac{15}{7} \geq \frac{1}{2}(\psi(G(3,3,3) + G(0,1,1))$ so it is a generalization.

**Theorem 3.3.** Let $(X, G)$ be a complete $G$-metric space. Let $F : X \times X \to X$ be generalized $(\alpha, \psi)$-contractive map in two variables satisfying the following conditions.

(i) for all $(x,y),(u,v) \in X \times X$, we have

$$\alpha((x,y),(u,v),(u,v)) \geq 1$$

implies

$$\alpha((F(x,y), F(y,x),(F(u,v), F(v,u)), (F(u,v), F(v,u))) \geq 1$$

(ii) there exist $(x_0, y_0, y_0) \in X \times X \times X$ such that

$$\alpha((x_0,y_0),(F(x_0,y_0),(F(y_0,x_0)), (F(x_0,y_0), (F(y_0,x_0)) \geq 1$$

and

$$\alpha((F(y_0,x_0),(F(x_0,y_0),(F(y_0,x_0), (F(x_0,y_0))), (F(x_0,y_0), (F(x_0,y_0))))) \geq 1$$
(iii) $F$ is continuous.

then $F$ has a coupled fixed point. i.e., there exist $(x^*, y^*) \in X \times X$ such that $F(x^*, y^*) = x^*$ and $F(y^*, x^*) = y^*$.

Proof. Let $(Y, \delta)$ be a complete $G$-metric space with $Y = X \times X$ and 
\begin{align*}
\delta((x, y), (u, v), (s, t)) = G(x, u, s) + G(y, v, t) \quad \text{for all} \quad (x, y), (u, v), (s, t) \in Y.
\end{align*} 

By using (3.3.1) and (G4) we get
\begin{align*}
\alpha(x, y), (u, v), (u, v)G(F(x, y), F(u, v), F(u, v)) &\leq \frac{1}{2}\psi(\max\{G(x, u, v) + G(y, v, v), \\
&\frac{1}{2}(G(x, F(x, y), F(x, y)) + G(y, F(y, x), F(y, x)) + (G(u, F(u, v), F(u, v)) \\
&+ G(v, F(u, v), F(u, v))), \frac{1}{2}(G(x, F(u, v), F(u, v)) + G(y, F(v, u), F(v, u))) \\
&+ (G(u, F(x, y), F(x, y)) + G(v, F(y, x), F(y, x))))
\end{align*}

Similarly
\begin{align*}
\alpha((x, y), (u, v), (u, v))G(F(x, y), F(u, v), F(u, v)) &\leq \frac{1}{2}\psi(\max\{G(x, u, v) + G(y, v, v), \\
&\frac{1}{2}(G(x, F(x, y), F(x, y)) + G(y, F(y, x), F(y, x)) + (G(u, F(u, v), F(u, v)) \\
&+ G(v, F(u, v), F(u, v))), \frac{1}{2}(G(x, F(u, v), F(u, v)) + G(y, F(v, u), F(v, u))) \\
&+ (G(u, F(x, y), F(x, y)) + G(v, F(y, x), F(y, x))))
\end{align*}

Adding the inequalities (3.3.1) and (3.3.2) we get
\begin{align*}
\alpha((x, y), (u, v), (u, v))G(F(x, y), F(u, v), F(u, v)) &+ \alpha((v, u), (v, u), (y, x))G(F(v, u), F(u, v), F(u, v)) - \psi(\max\{G((x, y), (u, v), (u, v)), \\
&(\frac{1}{2}(G((x, y), F(x, y), F(x, y)) + G((y, x), F(y, x), F(y, x))) + (G((u, v), F(u, v), F(u, v)) \\
&+ G((v, u), F(u, v), F(u, v))), \frac{1}{2}(G((u, v), F(u, v), F(u, v)) + G((v, u), F(v, u), F(v, u))) \\
&+ (G((v, u), F(v, u), F(v, u)) + G((v, u), F(v, u), F(v, u)))) + \delta((u, v), (F(x, y), F(y, x)), (F(x, y), F(y, x)))
\end{align*} 

Choose $\beta : Y \times Y \to [0, \infty)$ is given by
\begin{align*}
\beta((x, y), (u, v), (u, v)) = \min\{\alpha((x, y), (u, v), (u, v)), \alpha((v, u), (v, u), (y, x))\}
\end{align*}

Define $T : Y \to Y$ by $T(x, y) = (F(x, y), F(y, x))$ Since $T$ is continuous and $G - \beta - \psi$ contractive mapping of equation (3.3.1)
\begin{align*}
\beta((x, y), (u, v), (u, v))\delta((F(x, y), (y, x)), (F(u, v), F(v, u)), (F(u, v), F(v, u))) \\
&\leq \psi(\max\{\delta((x, y), (u, v), (u, v)), \frac{1}{2}\delta((x, y), (F(x, y), F(y, x)), (F(x, y), F(y, x))) + \delta((u, v), (F(x, y), F(y, x)), (F(x, y), F(y, x)))
\end{align*}

That is
\begin{align*}
\beta((x, y), (u, v), (u, v))\delta(T(x, y), T(u, v)) &\leq \max\{\delta((x, y), (u, v), (u, v)), \\
&\frac{1}{2}\delta((x, y), (F(x, y), F(y, x)), (F(x, y), F(y, x)) + \delta((u, v), (F(x, y), F(y, x)), (F(x, y), F(y, x)))
\end{align*}

Therefore $T$ is a generalized $(\beta, \psi)$ contractive map.

We now show that $T$ is $\beta$-admissible.

Let $(x, y), (u, v) \in Y$ be such that
\begin{align*}
\beta((x, y), (u, v), (u, v)) &= \min\{\alpha((x, y), (u, v), (u, v)), \alpha((v, u), (v, u), (y, x))\} \\
&\geq 1
\end{align*} 

(3.3.4)
Since $\alpha((x, y), (u, v), (u, v)) \geq 1$ and $\alpha((v, u), (v, u), (y, x)) \geq 1$
Hence by (i),
\[
\beta(T(x, y), T(u, v), T(u, v)) = \beta((F(x, y), (y, x)), (F(u, v), F(v, u)), (F(u, v), F(v, u)))
\]
\[
= \min \{\alpha((F(x, y), (y, x)), (F(u, v), (v, u)), (F(u, v), F(v, u))),
\alpha((F(v, u), (u, v)), (F(v, u), (u, v)), (F(y, x), F(x, y)))\} \geq 1
\]
By using (3.3.4) and (i). Hence $T$ is $\beta-$admissible.
Now by (ii), there exists $(x, y) \in X \times X$ such that
\[
\beta((x_0, y_0), T(x_0, y_0), T(x_0, y_0)) = \beta((x_0, y_0), (F(x_0, y_0), F(y_0, x_0))
\]
\[
= \min \{\alpha((x_0, y_0), (F(x_0, y_0), F(y_0, x_0)), (F(x_0, y_0), F(y_0, x_0))),
\alpha((x_0, y_0), (F(x_0, y_0), F(y_0, x_0)), (F(x_0, y_0), F(y_0, x_0)))\} \geq 1
\]
We show that $T$ is continuous.
Let $(a, b) \in X \times X$. We have $T(x, y) = (F(x, y), F(y, x))$ for all $x, y \in X$.
Therefore
\[
\lim_{(x, y) \to (a, b)} T(x, y) = \lim_{(x, y) \to (a, b)} (F(x, y), F(y, x))
\]
\[
= \lim_{(x, y) \to (a, b)} (F(x, y), \lim_{(x, y) \to (a, b)} (F(y, x)))
\]
\[
= (F(a, b), (F(b, a)) = T(a, b)
\]
and hence $T$ is continuous. Hence $T$ has a fixed point $(x^*, y^*)$ in $Y$. Therefore by
Lemma 2.11 and $(x^*, y^*)$ is a coupled fixed point of $F$.

**Theorem 3.4.** Let $(X, G)$ be a complete $G$-metric space. Let $F : X \times X \to X$ be
generalized $(\alpha, \psi)$ contractive map in two variables satisfying the following conditions.

(i) for all $(x, y), (u, v) \in X \times X$, we have
\[
\alpha((x, y), (u, v), (u, v)) \geq 1 \implies \alpha((F(x, y), F(y, x)), (F(u, v), F(v, u)), (F(u, v), F(v, u))) \geq 1
\]

(ii) there exist $(x_0, y_0) \in X \times X$ such that
\[
\alpha((x_0, y_0), (F(x_0, y_0), F(y_0, x_0)), (F(x_0, y_0), F(y_0, x_0))) \geq 1
\]

(iii) if \{\(x_n\)\} and \{\(y_n\)\} are sequence of $X$ such that
\[
\alpha((x_n, y_n), (x_{n+1}, y_{n+1}), (x_{n+1}, y_{n+1})) \geq 1
\]
and $\alpha((y_{n+1}, x_{n+1}), (y_{n+1}, x_{n+1}, y_{n+1}), (y_{n+1}, x_{n+1})) \geq 1$
\{\(x_n\)\} and \{\(y_n\)\} are convergent to $x$ and $y$ respectively then
\[
\alpha((x_n, y_n), (x, y), (x, y)) \geq 1
\]
and $\alpha((y_n, x_n), (y, x), (y, x)) \geq 1$
then $F$ has a coupled fixed point. i.e there exist $(x^*, y^*) \in X \times X$ such that
\[
F(x^*, y^*) = x^* \text{ and } F(y^*, x^*) = y^*.
\]

**Proof.** In above Theorem 3.3 we prove (i) and (ii) conditions
Now to prove condition (iii) by using equation (ii)
let $(x_0, y_0, y_0) \in X \times X \times X$ such that
\[
\alpha((x_0, y_0), (F(x_0, y_0), F(y_0, x_0)), (F(x_0, y_0), F(y_0, x_0))) \geq 1
\]
and $\alpha((y_0, x_0), (F(y_0, x_0), F(x_0, y_0)), (F(y_0, x_0), F(x_0, y_0), F(y_0, x_0))) \geq 1$
(3.4.1)
We define the sequence \{(\(x_n\), \(y_n\)\}) $\in X \times X$ by
\[
x_{n+1} = F(x_n, y_n) \text{ and } y_{n+1} = F(y_n, x_n)
\]
Now $x_n = x_{n+1}$ and $y_n = y_{n+1}$ for some $n$ then $(x_n, y_n)$ is a coupled fixed point of $X$. 8
Now \( x_n \neq x_{n+1} \) and \( y_n \neq y_{n+1} \) then by equation (3.4.1) we have
\[
\alpha((x_0, y_0), (x_1, y_1), (x_2, y_2)) \geq 1
\]
and
\[
\alpha(F(x_0, y_0), F(y_0, x_0), (F(x_1, y_1), F(y_1, x_1), (F(x_1, y_1), F(y_1, x_1))) \geq 1
\]
i.e., \( \alpha(x_1, y_1), (x_2, y_2)) \geq 1 \) continuing this process, we get
\[
\alpha((x_n, y_n), (x_{n+1}, y_{n+1}), (x_{n+1}, y_{n+1})) \geq 1
\]
Now
\[
\begin{aligned}
\alpha((x_{n-1}, y_{n-1}), (x_n, y_n)) & \leq \frac{\psi}{2} \max \{\alpha(G(x_{n-1}, x_n), G(y_{n-1}, y_n)), \\
\frac{1}{2} \alpha(G(x_{n-1}, x_n), G(x_{n-1}, y_{n-1}) + \alpha(G(y_{n-1}, y_n), F(x_{n-1}, x_n), F(x_{n-1}, y_{n-1}))
\end{aligned}
\]
and so on.

Similarly,
\[
\begin{aligned}
G(F(x_{n-1}, y_{n-1}), F(y_{n-1}, y_{n-1})) & \leq \frac{\psi}{2} \max \{\alpha(G(x_{n-1}, x_n), G(y_{n-1}, y_n), G(x_{n-1}, x_n), F(y_{n-1}, x_{n-1}))
\end{aligned}
\]

On using the notation of \( \beta \) given in the proof of Theorem 3.3, we have
\[
\beta((x_n, y_n), (x_{n+1}, y_{n+1})) = \min \{\alpha((x_n, x_{n-1}), (x_0, x_0), (x_n, y_n)), \\
\alpha((y_n, y_{n-1}), (x_n, x_n), (y_{n-1}, x_{n-1})), 1 \} \geq 1.
\]

by using (3.4.1), (3.4.4) and (3.4.5), we have
\[
\begin{aligned}
G(F(x_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1})) & \leq \beta(x_{n-1}, x_{n-1}, (x_{n-1}, x_{n-1}))G(F(x_{n-1}, x_{n-1}), F(x_{n-1}, x_{n-1}))
\end{aligned}
\]
and
\[
\begin{aligned}
G(F(x_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1})) & \leq \beta(x_{n-1}, x_{n-1}, (x_{n-1}, x_{n-1}))G(F(x_{n-1}, x_{n-1}), F(x_{n-1}, x_{n-1}))
\end{aligned}
\]

It follows that \( \delta((x, y), (x, y), (x, y), (x, y), (x, y), (x, y), (x, y), (x, y)) \) and
\[
\begin{aligned}
\delta((x, y), (x, y), (x, y), (x, y), (x, y), (x, y), (x, y), (x, y)) \to \infty
\end{aligned}
\]
so that \( G(x, x_{n+1}, x_{n+1}) \to 0 \) and \( G(y, y_{n+1}, y_{n+1}) \to 0 \) as \( n \to \infty \).

Therefore, \( \delta((x, y), (x, y), (x, y), (x, y)) \) and
\[
\begin{aligned}
\delta((x, y), (x, y), (x, y), (x, y)) \to \infty
\end{aligned}
\]
Let $\epsilon > 0$ be given, Since $\sum_{n=1}^{\infty} \psi^n(\delta((x_0, y_0), (x_1, y_1), (x_1, y_1))) < \infty$
there exist $N \in \mathbb{Z}^+$ such that
$\sum_{n=1}^{\infty} \psi^n(\delta((x_0, y_0), (x_1, y_1), (x_1, y_1))) < \epsilon$ for all $n \geq N(\epsilon)$.
Now we show that $\{x_n\}$ is a Cauchy sequence in $X$.
Let $m, n \in \mathbb{Z}^+$ with $m > n \geq N$.
$G(x_n, x_{n+k}, x_{n+k}) = G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \ldots + G(x_{m-1}, x_m, x_m)$
$\leq \psi^n(\delta((x_0, y_0), (x_1, y_1), (x_1, y_1))) + \ldots + \psi^{m-1}(\delta((x_0, y_0), (x_1, y_1), (x_1, y_1)))$
$\leq \sum_{n=1}^{\infty} \psi^n G(x_0, x_1, x_1) < \epsilon$

i.e., $G(x_n, x_{n+k}, x_{n+k}) < \epsilon$ for all $n + k, n \geq N$
Hence $\{x_n\}$ is a Cauchy sequence in $X$.
Since $X$ is complete, there exist $x \in X$ such that
$\lim_{n \to \infty} x_n = x$. Similarly we can prove that $\{y_n\}$ is a Cauchy sequence in $X$. 
Since $X$ is complete, there exist $y \in X$ such that $\lim_{n \to \infty} y_n = y$.
Thus, $\{x_n\}$ and $\{y_n\}$ are sequence in $X$ such that
\begin{align*}
\alpha((x_n, y_n), (x_{n+1}, y_{n+1}) & \geq 1, \\
\alpha((y_{n+1}, x_{n+1}), (y_{n+1}, x_{n+1})) & \geq 1, \\
\{x_n\} \to x \in X ~ \text{and} ~ \{y_n\} \to y \in X \text{ as } n \to \infty.
\end{align*}
By (iii) we get, $\alpha((x_n, y_n), (x, y)) \geq 1$ and $\alpha((y_n, x_n), (y, x)) \geq 1$ for all $n$.
Therefore $T$ and $\beta$ satisfy all the hypothesis of Theorem 3.4. Hence $T$ has a fixed point and $F$ has a coupled fixed point.

**Theorem 3.5.** In addition to the hypotheses of Theorem 3.4, if condition (H) holds, then uniqueness of coupled fixed point of $F$.

**Proof.** We show that $T$ and $\beta$ of Theorem 3.3 satisfy the hypotheses
Let $x, y, u, v \in X$. Then by using (H), we get
$\beta((x, y), (z_1, z_2), (z_1, z_2)) = \min\{\alpha((x, y), (z_1, z_2), (z_1, z_2)), \alpha((z_2, z_1), (z_2, z_1), (y, x))\}$
Similarly
$\beta((u, v), (z_1, z_2), (z_1, z_2)) = \min\{\alpha((u, v), (z_1, z_2), (z_1, z_2)), \alpha((z_2, z_1), (z_2, z_1), (v, u))\}$
Hence $T$ and $\beta$ satisfy the hypotheses of Theorem 3.3. $T$ has a unique fixed point and consequently by Lemma 2.11 and $F$ has a unique coupled fixed point.

The following is an example in support of Theorem 3.3.

**Example 3.6.** Let $(X, G)$ be a $G$-metric space, where $X = [0, 1]$ and
$G(x, y, z) = |x - y| + |y - z| + |z - x|$ for all $x, y, z \in X$.
We define $F : X \times X \to X$ by $F(x, y) = \frac{1}{2}xy$ for all $x, y \in X$.
We define $\alpha : X^2 \times X^2 \times X^2 \rightarrow X$ be such that
$\alpha((x, y), (u, v), (u, v)) = \begin{cases} 1 & \text{if } x \geq u, y \leq v \\ 0 & \text{otherwise.} \end{cases}$
Since $|xy - uv| \leq |x - u| + |y - v|$ holds for all $x, y, u, v \in X$. 

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Suppose \( x \geq u, y \leq v \) then
\[
\alpha((x, y), (u, v)(u, v))G(F(x, y), F(u, v), F(u, v)) = 1, G(u, F(u, v), F(v, u)) = 1, G(u, F(u, v), F(v, u)) = 1
\]
and \( F \) is continuous. we choose \[
\frac{1}{2}(G(x, F(x, y), F(x, y)) + G(y, F(y, x), F(y, x))) + (G(u, F(u, v), F(v, u)))
\]
and \( \frac{1}{2}(G(v, F(v, u), F(v, u))) = 1 \). Hence, hypotheses of Theorem 3.3 satisfy. Then there exist coupled fixed point in \( F \). In this case (1.0) and (0,1) are the coupled fixed point of \( F \).

This example is not necessary for this Theorem 3.4 and it has unique, it is sufficient condition for (H) condition.

**Example 3.7.** Let \((X, G)\) be a \(G\)-metric space, where \(X = \mathbb{R}\) and \(G(x, y, z) = |x - y| + |y - z| + |z - x|\) for all \(x, y, z \in X\).

We define \(F : X \times X \rightarrow X\) by \(F(x, y)\) = \(\frac{|x^2 - y^2|}{8}\) if \(x, y \in [0, 1]\)

We define \(\alpha : X^2 \times X^2 \times X^2 \rightarrow X\) be such that
\[
\alpha((x, y), (u, v), (u, v)) = \begin{cases} 
1 & \text{if } x, y, u, v \in [0, 1] \\
0 & \text{otherwise.}
\end{cases}
\]

Clearly \(F\) is not continuous at \((1,1)\) and \(F\) is generalized \((\alpha, \psi)\)-contractive map (i.e., \(F\) satisfy equation(3.1)), with \(\psi(t) = \frac{t}{2}\) for all \(t > 0\). In fact, for all \(x, y, u, v \in [0, 1]\) then
\[
\alpha((x, y), (u, v)(u, v))G(F(x, y), F(u, v), F(u, v)) = 2[F(x, y) - F(u, v)]
\]
and hence both side tends to zero. so that (3.4) holds for all \((x, y), (u, v) \in X \times X\) clearly (i) hold, we choose \(x_0 = \frac{1}{2}\) and \(y_0 = \frac{1}{2}\) it hold (ii).

Let \(x_n\) and \(y_n\) are sequence of \(X\) such that
\[
\alpha((x_n, y_n), (x_{n+1}, y_{n+1}), (x_{n+1}, y_{n+1})) \geq 1 \Rightarrow \{x_n\}, \{y_n\} \text{ are sequence in } [0,1]
\]
similarly \(\alpha((y_{n+1}, x_{n+1}), (x_{n+1}, x_{n+1}), (x_{n+1}, x_{n+1})) \geq 1 \Rightarrow \{y_n\}, \{x_n\} \text{ are sequence in } [0,1]
Let \( \{x_n\} \to x \) and \( \{y_n\} \to y \). Since \([0,1]\) is closed we have \( x, y \in [0,1] \). Therefore \( \alpha((x_n, y_n), (x, y), (x, y)) \geq 1 \) and \( \alpha((y, x), (y, x), (y_n, x_n)) \geq 1 \) so that \((iii)\) holds. Therefore \( F, \alpha \) and \( \psi \) satisfy all the hypotheses of Theorem 3.4 and \((0,0)\) is a coupled fixed point of \( F \).

Acknowledgement: This research is supported by the project (MRP-4507/14/SERO/UGC). The first author is thankful to UGC.

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