A new Weighted Exponential Distribution and its Applications on Waiting Time Data

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Abstract- Exponential distribution is one of the most useful distribution real life data. In this paper a new weighted exponential distribution is derived. The resultant distribution is positively skewed. The shape of the weighted exponential distribution is shown with graphs with different values of parameters and weights. Further the properties including cdf, moments, median, skewness and kurtosis, and Quantile function of the weighted exponential distribution is derived. Parameter of the weighted exponential distribution is estimated by using maximum likelihood and moment estimation methods. The memory less property for weighted exponential distribution has been proved. Reliability measures are also developed for the newly derived distribution. Finally the applications of the weighted exponential distribution are discussed on a waiting time data.
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Index Terms- WED, MLE, MOM, weighted distributions, reliability measure.

I. INTRODUCTION

This article provide a newly derived distribution named WED, its properties and its applications on a waiting time data set. Before discussing the derived results, a few sentences about background of the area are presented.

The sampling units cannot be selected with equal probability in observational studies because there exists no well defined sampling plan particularly in observational studies of human life, wildlife, insect, plant or fish population. In these populations, the recorded observations on individuals are biased and will not have the original distribution because this is not possible to prepare a list of all the units in the population and select a sample with equal or known probabilities. The researchers are not able to draw a random sample from a population under study, so, non-randomness or biasedness arises in the data. The weighted distributions arise when the recorded observations are not generated randomly. A unifying approach for correction of biases that exist in unequally weighted sample data is provided by weighted distribution theory. Probability distribution function (pdf) of the weighted distributions is

$$f(x;\theta)=\frac{w(x)f_{\circ}(x;\theta)}{E[w(x)]} \tag{1.1}$$

Where $w(x)>0$, is weight function, and $f_{\circ}(x;\theta)$ is parent distribution.

The weighted distributions introduced earlier by Fisher (1934) who studied the effect of ascertainment upon estimation of gene frequencies. Warren (1975) applied these distributions in forest product research. Patil and Rao (1977, 1978) surveyed the statistical applications of moment distributions, particularly for the analysis of data relating to human populations and ecology. Rao (1985) pointed out the situations in which moment distributions arise and discussed the analysis of data related to human populations and natural science. Kochar and Gupta (1987) derived the properties of moment distributions in comparison with the original distributions for positive random variables and formulated the bounds on the moments of moment distributions. Jain et al. (1989) derived the reliability measures of life distributions and established the relationships between moment distributions and classes of life distributions. Skolimowska (2006) described the preservation of stochastic orders and classes of life distribution under weighting. Dara and Ahmad (2011) developed some weighted (moment) distributions along with their reliability measures.

II. WEIGHTED EXPONNETIAL DISTRIBUTION (WED)

If X~ $Exp(\lambda)$ with pdf

$$f(x;\lambda)=\lambda e^{-\lambda}, \qquad x>0, \lambda>0. \tag{2.1}$$

Then by using the weight function $w(x)=e^{\omega x}, where \quad 0<\alpha<1,$ the pdf of the weighted exponential distribution (WED) is

$$f(x;\lambda)=(\lambda-\omega)e^{-(\lambda-\omega)x}, \qquad x>0, \quad \lambda>0, \lambda>\omega, \qquad 0<\omega<1. \tag{2.2}$$

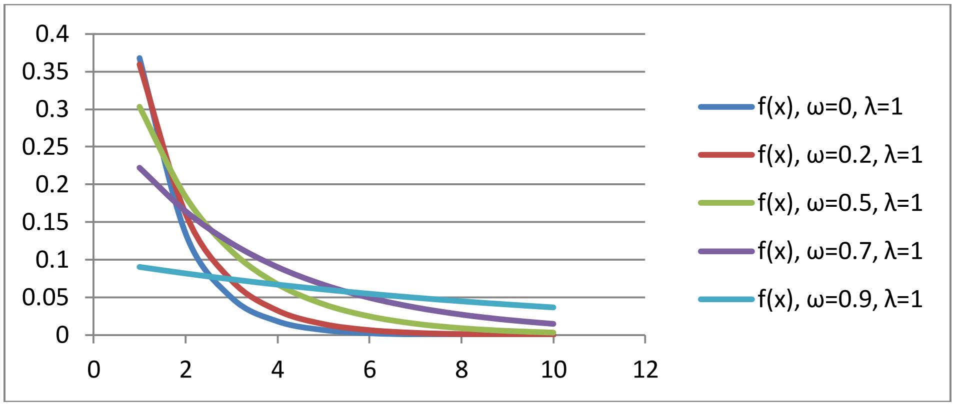

Fig1. Pdf plod of WED for $\lambda = 1$ and different values of ω.

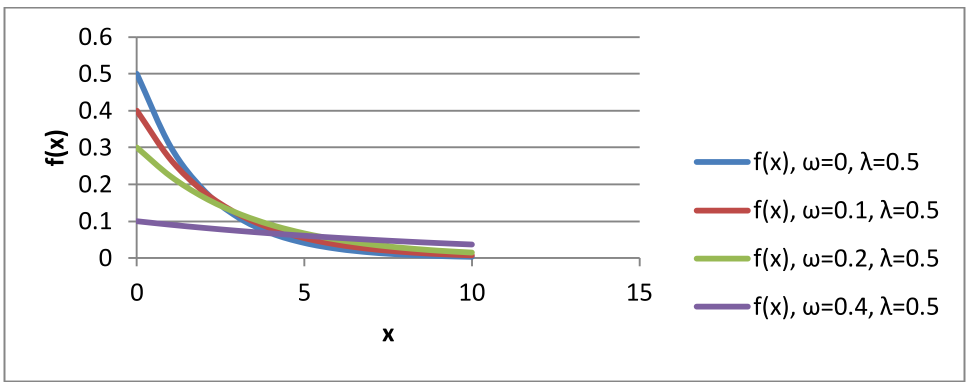

Fig2. Pdf plot of WED for $\lambda = 0.5$ and different values of ω.

The cumulative distribution function of the WED is

$$F(x) = 1 - e^{-(\lambda-\omega)x}, \qquad x > 0, \quad \lambda > 0, \quad 0 < \omega < 1. \tag{2.3}$$

III. PROPERTIES OF THE WEIGHTED EXPONENTIAL DISTRIBUTION

The rth moments of the WED are

$$\mu_r' = \frac{r\Gamma(r)}{(\lambda-\omega)^r}, \qquad \lambda > \omega. \tag{3.1}$$

The first four raw moments of the WED are

$$Mean = \mu_1' = \frac{1}{(\lambda-\omega)}, \qquad \lambda > \omega. \tag{3.2}$$

$$\mu_2' = \frac{2}{(\lambda-\omega)^2}, \quad \lambda > \omega, \quad \mu_3' = \frac{6}{(\lambda-\omega)^3}, \quad \lambda > \omega, \; \mu_4' = \frac{24}{(\lambda-\omega)^4}, \quad \lambda > \omega. \tag{3.3}$$

The mean moments of the WED are

$$Variance = \mu_2 = \frac{1}{(\lambda-\omega)^2}, \quad \lambda > \omega. \tag{3.4}$$

$$\mu_3 = \frac{2}{(\lambda-\omega)^3}, \quad \lambda > \omega, \qquad \mu_4 = \frac{9}{(\lambda-\omega)^4}, \quad \lambda > \omega, \tag{3.5}$$

The skewness and kurtosis of the WED are

$$\beta_1 = 4 \quad and \quad \gamma_1 = \sqrt{\beta_1} = 2. \tag{3.6}$$

$$\beta_2 = 9 \quad and \quad \gamma_2 = \beta_2 - 3 = 6. \tag{3.7}$$

The skewness and kurtosis of the WED are independent of parameters. It can be seen that the WED is positively skewed and leptokurtic.
Mode of the WED is "0" and Median of the WED is

$$m(X) = \frac{\ln(2)}{(\lambda-\omega)} < E(X), \quad \lambda > \omega. \tag{3.8}$$

Thus the absolute difference between the mean and median is

$$|E(X) - m(X)| = \frac{1-\ln(2)}{(\lambda-\omega)} \tag{3.9}$$

Quantile function of the WED is

$$x = \frac{-\ln(1-p)}{(\lambda-\omega)}, \qquad \lambda > \omega. \tag{3.10}$$

The Quantiles can be fine by using above ( ). Let $(1-p) = 0.5$, then it is median and $(1-p) = 0.25$, then it is lower quartile and $(1-p) = 0.75$, then it is upper quartile etc.

Memory less property: If X follows the WED the prove that $P(X > s+t/X > t) = P(X > s)$.

Proof: Let $P(X > s+t/X > t) = \frac{P(x > s+t/X > t)}{P(X > t)}$

$$P(X > s+t/X > t) = \frac{P(X > s+t)}{P(X > t)}$$
$$= \frac{e^{-(\lambda-\omega)(s+t)}}{e^{-(\lambda-\omega)t}}$$
$$= e^{-(\lambda-\omega)s}$$
$$= P(X > s).$$

IV. RELIABILITY MEASURES

The Survival function of the WED is

$$S(x) = e^{-(\lambda-\omega)x} \tag{4.1}$$

The hazard function of the WED is

$$h(x) = (\lambda-\omega), \qquad \lambda > \omega. \tag{4.2}$$

The cumulative hazard rate of the WED is

$$\Lambda(x) = (\lambda-\omega)x, \qquad \lambda > \omega. \tag{4.3}$$

The mean residual life of the WED is

$$\mu(x) = \frac{1}{(\lambda - \omega)}, \qquad \lambda > \omega. \tag{4.4}$$

Hence

$$\mu(0) = \mu = \mu_1' = \frac{1}{(\lambda - \omega)} \tag{4.5}$$

Entropy of the WED is

$$H(x) = 1 - \ln(\lambda - \omega), \qquad \lambda > \omega. \tag{4.6}$$

V. ESTIMATION

In this section the parameters of the WED are estimated by maximum likelihood estimation (MLE) and method of moments (MOM).

Maximum Likelihood Estimator

The MLE of WED for λ is

$$\hat{\lambda} = \frac{1}{\bar{x}} + \omega, \qquad 0 < \omega < 1. \tag{5.1}$$

Method of Moment Estimator

The MOM of WED for λ is

$$\tilde{\lambda} = \frac{1}{\bar{x}} + \omega, \qquad 0 < \omega < 1. \tag{5.2}$$

It can be seen that the MLE and MOM is not unbiased estimator.

VI. APPLICATIONS

The fitting of the WED to data-set relating to waiting times (in minutes) of 100 bank customers reported by Ghitany et al (2008) have been presented in the following table.

Table 1: Goodness of fit for WED on waiting times (in minutes) of 100 bank customers

Waiting time (in minutes)	**Observed frequencies**	**Expected frequencies Weighted exponential distribution**
0-5	30	40.25
5-10	32	24.049
10-15	19	14.37
15-20	10	8.586
20-25	5	5.13
25-30	1	3.065
30-35	2	1.831
35-40	1	1.095
Total	100	98.34
Estimation of parameters		$\lambda = 0.60$, $\omega = 0.5$
Chi-square value		7.628597
d.f		3
p-value		0.054347

VII. CONCLUSION

The newly derived WED is positively skewed and can be used in life data analysis. Various properties of the WED are derived in this paper. Reliability measures of the WED are derived. Like exponential distribution the WED has a constant failure rate. The MLE and MOM estimators are derived and they are biased. The WED is applied on waiting times (in minutes) of 100 bank customers and it is proved that the distribution is showing good fit for the data with chi-square value = 7.628597 and p-value = 0.054347.

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