ISSN 2250-3153

Ky Fan's Best Approximation Theorem in Hilbert space

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Abstract- The aim of this paper is to prove a fixed point theorem using semicontractive mapping a well-known result of Ky Fan in Hilbert space.

Keywords - Fixed point theorem, Ky Fan's best approximation theorem, Semicontractive mapping.

AMS (2010) Subject Classification - 47H10, 54H25.

I. INTRODUCTION

Fixed point theory has always been existing itself and its applications in new areas. The theory of approximation also played an important role. Ky Fan one of the great mathematician establish an existing theorem in 1969, which was known as Ky Fan's best approximation theorem which has been of great importance in nonlinear analysis, minimax theory and approximation theory. Several interesting fixed point theorems have been proved by using Ky Fan's best approximation theorem. This approach helps to find fixed point theorems under different boundary conditions. Most of the fixed point theorems are given for self maps that are for a function with domain and range are the same. In case a function does not have the same domain and range then we need a boundary condition to guarantee the existence of fixed point.

Let X be a normed linear space and K be a nonempty subset of X. Let $T : K \to X$ be a function. We look for an x in K that satisfies the following equation

$$\|x - T(x)\| = d(T(x), K) = \inf\{\|y - T(x) : y \in K\}\| --------(A)$$

If a solution x in K exists, it is called a best approximation for $T(x)$. We note that $x \in K$ is a solution of (A) if and only if x is a fixed point of $Q_K \, \mathrm{o} T$ where Q_K is the metric projection on K. We refer to Carbone [2], Caristi [3], Cheney [4], Furi Et al. [6], Kuratowski [8], Nussbaum [10], Park [11], Schoneberg [14], Singh and Watson [15]. In this paper fixed point theorem has been established using the concept of semicontractive mapping which generalized the result of some standard result.

II. PRELIMINARIES

Lemma 2.1 [1] Suppose that H be a Hilbert space and K be a nonempty closed convex subset of X. A function $T : K \to H$ is called semicontractive if there exists a mapping D of $H \times H \to K$ such that:

(i) for each fixed x in K $T(x) = D(x, x),$
(ii) for each fixed x in K , $D(x, \cdot)$ is compact,
(iii) for each fixed x in K , $D(\cdot, x)$ is nonexpansive.

Corollary 2.2[13] Suppose that K be a closed bounded and convex subset of H and suppose $T : K \to H$ be a semicontractive. Then there exists a $y \in K$ such that $\|y - T(y)\| = d(Ty, K)$.

Definition 2.3[7] Suppose that K be a subset of a Hilbert space H for each $x \in K$. Let the inward set of K at $x, I_K(x)$ be defined by $I_K(x) = \{x + r(t - x) : t \in K, r > 0\}.$

A mapping $T : K \to H$ is said to be inward if for each $x \in K,$ $T(x)$ lies in $I_K(x)$ and it is weakly inward if $T(x)$ lies in $\overline{I_K}(x)$.

Theorem 2.4 [9] Suppose that H be a Hilbert space and K be a nonempty closed convex subset of H, T a continuous semicontractive map of K into H. Let either $(\mathrm{I}-\mathrm{Q}_K \mathrm{o}\mathrm{T})(\mathrm{K})$ is closed in H or $(\mathrm{I}-\mathrm{Q}_K \mathrm{o}\mathrm{T})(\mathrm{clco}(\mathrm{Q}_K \mathrm{o}\mathrm{T}(K)))$, where Q_K, is the proximity map of H into K. If $T(K)$ is bounded then there exists a point v in K such that $\|v-T(v)\| = d(T(v), K).$

III. MAIN RESULTS

Theorem 3.1 Suppose that H be a Hilbert space and K be a nonempty closed convex subset of H and T be a continuous semicontractive map of K into H. Let either $(I-Q_K \mathrm{o}T)(clco(Q_K \mathrm{o}T(K)))$ or $(I-Q_K \mathrm{o}T)(K)$ is closed in H where Q_K is the proximity map of H into K. Suppose $T(K)$ is bounded and T has a fixed point in K if and only if it satisfies one of the conditions below:

[1] $\exists$ y in $I_K(x) = \{x + t(z-x):$ for some $z \in K,$ some $t > 0\}$ such that $\|y-T(x)\| < \|x-T(x)\|,$ for $x \in K$ with $x \neq T(x).$

Proof: Consider that T satisfies condition. By using theorem 2.4 $\exists$ a point v in K such that $\|v-T(v)\| = d(T(v), K).$ If $v \neq T(v),$ then $\exists$ a y in $I_K(v)$ such that $\|y-T(v)\| < \|v-T(v)\|.$ If $y \in K,$ which is a contradiction for supposition of v. Hence $y \notin K,$ and $\exists$ a $z \in K,$ such that $y = v + t(z-v)$ for some $t > 1.$

$$\text{i.e. } z = \frac{1}{t}y + (1-\frac{1}{t})v = (1-\lambda)y + \lambda v \text{ where } \lambda = 1-\frac{1}{t}, 0 < \lambda < 1.$$

Hence

$$\|z-T(v)\| = \|(1-\lambda)y + \lambda v - T(v)\| \leq (1-\lambda)\|y-T(y)\| + \lambda\|v-T(v)\|$$

$$< (1-\lambda)\|v-T(v)\| + \lambda\|v-T(v)\| = \|v-T(v)\|$$

Which contradicts the supposition of v.

Hence $v = T(v)$

[2] There is a number α real or complex depending on the vector space X respectively. For each $x \in K$, such that $|\alpha| < 1$ and $\alpha x + (1-\alpha)T(x) \in K.$

Proof: Consider that T satisfies condition. Using theorem2.4 $\exists$ point v in K such that $\|v-T(v)\| = d(T(v), K).$ Let T has no fixed point in K, then $0 < \|v-T(v)\|$. For point v, there is a number α such that $|\alpha| < 1$ and $\alpha v + (1-\alpha)T(v) = x \in K$.

Therefore $0 < \|v-T(v)\| = d(T(v), K) \leq \|x-T(v)\| = |\alpha|\|v-T(v)\| < \|v-T(v)\|$

Which our supposition. Hence T has a fixed point in K.

[3] If $v = Q_K \mathrm{o}T(v),$ where v be any point on the boundary of K, then v is a fixed point of T.

Proof: Consider that T satisfies condition. Using theorem 2.4 $\exists$ a point v in K such that $\|v-T(v)\| = d(T(v), K).$ If $T(v) \in K$, then $d(T(v), K) = 0$ and v is a fixed point of T. If $T(v) \notin K$ then from $\|T(v) - Q_K \mathrm{o}T(v)\| = d(T(v), K) = \|T(v) - v\|,$ and the uniqueness of the nearest point, $Q_K \mathrm{o}T(v) = v$. Implies that v lies on the boundary of K, which contradicts our supposition. Hence v is a fixed point of T.

[4] $\forall\ x \in K,\ T(x) \in clI_K(x),$ i.e. T is weakly inward.

Proof: Consider that T satisfies condition. $\forall x \in K, T(x) \in clI_K(x)$. If $x \neq T(x)$ then there exists a point y in $I_K(x)$ such that $y \in B(T(x), \frac{\|x-T(x)\|}{2})$, where $B\{T(x), \frac{\|x-T(x)\|}{2}\},$ is an open ball with centre $T(x)$ and radius $\frac{\|x-T(x)\|}{2}$, Therefore $\|y-T(x)\| < \|x-T(x)\|.$
Hence T has a fixed point in K.

[5] $\forall\ x$ On the boundary of K, $\|T(x)-y\| \le \|x-y\|$ for some y in K.

Proof: Proof of this condition is similar to the proof of condition (1).

Corollary 3.2 Suppose that H be a Hilbert space and K be a nonempty closed convex subset of a Hilbert space H and T be a continuous 1-setcontraction map of K into H. If $T(K)$ is bounded and T satisfies any one of the five conditions of Theorem 3.1.Then T has a fixed point in K.

Corollary 3.3 Suppose that H be a Hilbert space and K be a closed convex subset of Hilbert space H. Suppose that $T: K \to H$ be semicontractive mapping with bounded range such that for each $x \in \partial K$, $\|Tx-y\| \le \|x-y\|$, for some $y \in K.$ Then T has a fixed point.

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