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Mean Value Prediction of the Biased Estimators

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Abstract- In this paper we formulate Stein–minimax type estimators and compare the performance properties with other estimators in case of mean value predictions. When the model is estimated by ordinary least squares it has been observed that least squares predicted is unbiased while minimax and Stein-minimax predictors are biased. The superiority conditions of the estimators have been derived by assuming error distribution to be non-normal.

Index Terms- Linear regression model, Minimax estimator, Stein-minimax estimator, Mean value prediction.

I. Introduction

Prediction is an important aspect of relationship analysis in any scientific study. It not only reflects the adequacy of underlying model but also assist in making a suitable choice among the competing models. Generally, predictions of study variable in linear models are obtained either for actual values or for average-values but not for both simultaneously. Section-2 describes the model specification and the predictors. Section-3 deals with the properties of the estimators. We have derived the expression for the prediction of mean value of the study variable for minimax and Stein-minimax estimator separately and their results are presented in the form of theorems in Section-4. In Section-5 comparative study have been made. Proof of the theorems are given in Section-6.

II. Model Specification and Estimators

Let the true linear regression model is

(2.1) $$Y = X\beta + \sigma u$$

Where Y is a (n ×1) vector of n observations on the variables to be explained, X is a (n×p) matrix of n observations on p explanatory variables, β is a $(p \times 1)$ vector of regression coefficients, σ being unknown scalar and u is a $(n \times 1)$ vector of disturbances with mean zero, variance unity and measures of skewness and kurtosis are γ_1 and (γ_2 +3) respectively.

Application of least squares to (2.1) yield the ordinary least squares estimator as

(2.2) $$b_0 = \left(X^1 X\right)^{-1} X'Y$$

which is well known to be the best linear unbiased estimator of β, having variance- co-variance matrix.

(2.3) $$V(b_0) = \sigma^2 \left(X^1 X\right)^{-1}$$

The Stein rule estimator of the regression co-efficient from (2.1) is given by

(2.4) $$b_s = \left[1 - K \frac{\left(Y - Xb_0\right)'\left(Y - Xb_0\right)}{b_o' X'Xb_0}\right] b_0$$

where K is any positive non-stochastic scalar characterizing the estimation.

The MILE proposed by Kuks and Olman (1971, 1972) and Kuks (1972) is

(2.5) $$b^* = D^{-1} X' W^{-1} Y$$

where

(2.6) $$D = \left(\alpha^{-1}\sigma^2 T + C\right)$$

Using above defined quantities, we obtain minimax estimator

(2.7) $$b^* = \beta + \sigma\left(X^1 W^{-1} X\right)^{-1} X^1 W^{-1} u - \sigma^2 \frac{T}{\alpha}. \left(X^1 W^{-1} X\right)^{-1} \beta - \sigma^3 \frac{T}{\alpha} X^1 W^{-1} u \left(X^1 W^{-1} X\right)^{-2}$$

The Stein- minimax regression coefficient of β is obtained by interchanging b_0 by b^* in (2.4) and is given by

(2.8) $$b_s^* = \left[1 - K \frac{\left(Y - Xb^*\right)^1 \left(Y - Xb^*\right)}{b^* X^1 X b^*}\right] b^*$$

Using above defined quantities, we obtain Stein–minimax estimator

(2.9 $$b_s^* = \beta + \sigma\left(X^1 W^{-1} X\right)^{-1} X^1 W^{-1} u$$

$$-\sigma^2\left[\begin{array}{l}\frac{T}{\alpha}\left(X^1W^{-1}X\right)^{-1}\beta\\ +K\frac{u^1Bu}{B^1X^1X\beta}\beta\end{array}\right]$$

$$-\sigma^3\left[\begin{array}{l}\frac{T}{\alpha}X^1W^{-1}u\left(X'W^{-1}X^{-2}\right)+K\frac{u^1Bu}{\beta_1X^1X\beta}\\ \left(X^1W^{-1}X\right)^{-1}X^1W^{-1}u\end{array}\right.$$

$$+\frac{2K}{\alpha}\beta^1\left[\left(X^1X\right)^{-1}-\left(X^1W^{-1}X\right)^{-2}\right]$$

$$\left.\frac{TX^1u\beta}{\beta^1X^1X\beta}-2K\frac{u^1Bu}{\beta^1X^1X\beta}\beta^1X^1u.\beta\right]$$

$$+\frac{\sigma^4}{\beta'X'X\beta}\left[\begin{array}{l}K\frac{U'BU}{\beta'X'X\beta}\beta\left(\begin{array}{l}X'W^{-1}X.X'U\\ -2\frac{T}{\alpha}\beta'X'X\beta\end{array}\right)\\ +\frac{2K}{\beta'X'X\beta}\end{array}\right.$$

.

$$\left\{\beta'X'u\left(\begin{array}{l}u'Bu\left(X'W^{-1}X\right)^{-1}X'W^{-1}u+\\ \frac{2}{\alpha}\beta'\left(\left(X'X\right)^{-1}-\left(X'W^{-1}X\right)^{-2}TX'u\beta\right)\end{array}\right)\right\}$$

$$+\frac{K}{\alpha}\left(X'W^{-1}X\right)^{-1}\left\{\beta\left(Tu'Bu-\frac{\beta'}{\alpha}\left(X'W^{-1}X\right)^{-1}\left(X'X\right)^2\right)\right\}$$

$$+2K\beta'\left\{\left(X'X\right)^{-1}-\left(X'W^{-1}X\right)^{-2}\right\}TXU'.X'W^{-1}u\Big]$$

The Predictors

To study the performance properties of these estimators for prediction purposes, let us postulate the following prediction vectors.

(2.10) $\hat{T}b^* = Xb^*$

(2.11) $Tb_s^* = Xb_s^*$

where b^*, b_s^* are defined earlier.

III. Properties of Estimator

In order to study the properties of the estimator, we observe that the exact expressions for the bias vectors mean squared error matrices and risk functions of the least squares estimators from model (2.1) can be easily obtained. However, it is not so with the minimax linear estimator. Therefore, in order to derive the expressions for the bias vectors, mean squared error matrices and risk functions, we introduce the following notations:

(3.1) $P_X = X(X'X)^{-1}X'$

(3.2) $M = 1 - P_X$

Here it is easily seen that

$$M = M' \;:M'.M = M\,;\,MX = X'M = 0$$

Substituting (2.1) in (2.2), we observe that b_o is an unbiased estimator of β with variance – covariance matrix as

(3.3) $V(b_o) = E(b_o-\beta)'(b_o-\beta) = \sigma^2\overline{P}_X$

Where $\overline{P}_X = \left[\,1 - X(X'X)^{-1}X'\,\right]$

The distributional assumption regarding the disturbances does not have any effect on the properties of least squares estimator. Clearly, the minimax estimator and Stein-minimax estimator is a biased estimator.

IV. Predictions

In this section, we have considered prediction of mean value of the study variable separately.

Mean Value Prediction

When the interest lies in prediction of mean value of the study variable, $\hat{T}_o$ are found to be unbiased but $Tb^*\,and\,Tb_s^*$ are found to be biased.

(4.1) $E\left[\hat{T}_o - E(Y)\right] = E\left[X\beta + \sigma PXu - X\beta\right] = 0$

Theorem (4.1): When disturbances are small, the bias vector and predictive risk of predictor $\hat{T}b^*$ up to the order of our approximation $0(\sigma^4)$ are given by

(4.2) $PB\left(\hat{T}b^*\right) = -\sigma^2\frac{T}{\alpha}X\left(X'W^{-1}X\right)^{-1}\beta$

(4.3) $PR\left(\hat{T}b^*\right) = \sigma^2 n - \sigma^4\left[\begin{array}{l}2\frac{T}{\alpha}\left(X'W^{-1}X\right)^{-1}+\\ \frac{T'T}{\alpha^2}\left(X'W^{-1}X\right)^{-1}\beta'X'X\beta\end{array}\right]$

Theorem (4.2): When disturbances are small, the bias vector and predictive risk of predictor $\hat{T}b_s^*$ up to the order of our approximation $0(\sigma^4)$ are given by

$$PB\left(\hat{T}b_s^*\right) = -\sigma^2 \begin{bmatrix} \frac{T}{\alpha} X\left(X'W^{-1}X\right)^{-1}\beta + \\ K.\frac{1}{\beta'X'X\beta} trB.X\beta \end{bmatrix} \tag{4.4}$$

$$-\sigma^3 \begin{bmatrix} \frac{K}{\beta'X'X\beta} X\left(X'W^{-1}X\right)^{-1} X'W^{-1} \\ \gamma_1(I_n * B)e - 2K\gamma_1(I_n * B)e \end{bmatrix}$$

$$+\frac{\sigma^4}{\beta'X'X\beta}\begin{bmatrix} \frac{K}{\beta'X'X\beta} X\beta.X'W^{-1}X.X' \\ \gamma_1(I_n * B)e - 2K\frac{T}{\alpha}trB \end{bmatrix}$$

$$+\frac{2K}{\beta'X'X\beta}\begin{Bmatrix} \beta'X'X\left(X'W^{-1}X\right) - X'W^{-1} \\ \left(\gamma_2(I_n * B) + tr(B) + 2B\right) \end{Bmatrix}$$

$$+\frac{KT}{\alpha}\left(X'W^{-1}X\right)^{-1}X\beta.tr(B) - \frac{K}{\alpha^2}\left(X'W^{-1}X\right)^{-1}$$

$$\beta\left(\left(X'W^{-1}X\right)^{-1}\left(X'X\right)^2\right)$$

$$+2K\beta'\left\{\left(X'X\right)^{-1} - \left(X'W^{-1}X\right)^{-2}\right\}TX.XX'W^{-1}\Big]$$

$$PR\left(Tb_s^*\right) = \sigma_n^2 - \sigma^4 \begin{bmatrix} 2\frac{T}{\alpha}\left(X'W^{-1}X\right)^{-1} - \\ \frac{T'T}{\alpha^2}\left(X'W^{-1}X\right)^{-1}\beta'X'X\beta' \end{bmatrix} \tag{4.5}$$

$$-\frac{K^2}{\beta'X'X\beta}\left\{\gamma_2 trB\left(I_n * B\right) + tr(B).B + 2trB\right\}\Big]$$

From (4.2) and (4.4), it is clearly seen that predictors $Tb^* and \hat{T}b_s^*$ are biased. This is in contrast with the unbiasedness of T_O.

V. A Comparison

On comparing the risk associated with $\left(\hat{T}_0\right)$ OLS estimator and $\left(T_b^*\right)$ minimax estimator we observe that

$$R\left(\hat{T}_0\right) - R\left(\hat{T}_b^*\right) = \sigma^4 \begin{pmatrix} 2\frac{T}{\alpha}\left(X'W^{-1}X\right)^{-1} + \frac{T'T}{\alpha^2} \\ \left(X'W^{-1}X\right)^{-1}\beta'X'X\beta \end{pmatrix} \tag{5.1}$$

By observing the above expression, we find that the predictor based on minimax linear estimator performs better than the predictor based on ordinary least square estimator.

* On comparing the risk associated with $\hat{T}_0(\text{OLS})$ and $\left(Tb_s^*\right)$ Stein-minimax estimator, we observe that

$$Risk\left(\hat{T}_0\right) - Risk\left(\hat{T}_b^*\right) = \sigma^4 \Bigg[2\frac{T}{\alpha}\left(X'W^{-1}X\right)^{-1} \tag{5.2}$$

$$+\frac{T'T}{\alpha^2}\left(X'W^{-1}X\right)^{-1}\beta'X'X\beta - \frac{K^2}{\beta'X'X\beta}.$$

$$\left\{\gamma_2 B\left(I_n * B\right) + tr(B).B + 2trB\right\}\Big]$$

When the distribution of disturbances is leptokurtic or mesokurtic i.e. $\gamma_2 \geq 0$, the predictor based on Stein-minimax estimation procedure performs better than that based on ordinary least squares predictor. Hence we find that Stein-minimax estimator is more efficient than ordinary least square estimator.

* On comparing the risk associated with minimax estimator $\left(\hat{T}_b^*\right)$, and Stein-minimax estimator $\left(Tb_s^*\right)$, we observe that

(5.3)

$$Risk\left(\hat{T}_b^*\right) - Risk\left(\hat{T}b_s^*\right) = \sigma^4\left[\frac{K^2}{B'X'X\beta}\begin{Bmatrix} \gamma_2 trB\left(I_n * B\right) \\ + tr(B).B + 2trB \end{Bmatrix}\right]$$

Again, when the distribution of disturbances is leptokurtic or mesokurtic $\gamma_2 \geq 0$, it is easy to observe that the predictor based on Stein minimax estimator performs better than based on minimax estimator. Hence we find that Stein-minimax estimator is more efficient then minimax estimator.

VI. Proof of the Theorems

Theorem 4.1

Using (2.7), we get

$$\hat{T}_b^* = X\beta + \sigma\pi_1 + \sigma^2\pi_2 + \sigma^3\pi_3 \tag{6.1}$$

where

(6.2) $$\pi_1 = X\left(X'W^{-1}X\right)^{-1}X'W^{-1}u$$

(6.3) $$\pi_2 = -\frac{T}{\alpha}X\left(X'W^{-1}X\right)^{-1}\beta$$

(6.4) $$\pi_3 = -\frac{T}{\alpha}X\left(X'W^{-1}X\right)^{-2}X'W^{-1}u$$

We have

(6.5) $$E(Y) = X\beta$$

Using (6.5) and (6.1), we observe that

(6.6) $$\left[\hat{T}b^* - E(Y)\right] = \sigma\pi_1 + \sigma^2\pi_2 + \sigma^3\pi_3$$

where π_1, π_2 and π_3 defined in (6.2), (6.3), (6.4) respectively. The result (4.2) of bias vector associated with theorem (4.1) can be obtained from

(6.7) $$PB\left(\hat{T}b^*\right) = E\left(\hat{T}b^* - E(Y)\right)$$
$$= \sigma E(\pi_1) + \sigma^2 E(\pi_2) + \sigma^3 E(\pi_3)$$

where

$$E[\pi_1] = 0 : E[\pi_3]$$

Now, by taking expectation of

$$E\left(\pi_1{}'\pi_1\right);\ E\left[\pi_1{}'\pi_2\right];\ E\left[\pi_1{}'\pi_3\right]$$
$$E\left[\pi_1{}'\pi_3\right];\ E\left[\pi_2{}'\pi_1\right];\ E\left[\pi_3{}'\pi_1\right]$$

Utilizing these expectation in the expression

(6.8) $$PR\left(\hat{T}b^*\right) = E\left[\left\{\hat{T}b^* - E(Y)\right\}^1\left\{\hat{T}b^* - E(Y)\right\}\right]$$
$$= \sigma^2 E\left[\pi_1^1\pi_1\right] + \sigma^3 E\left[\pi_1^1\pi_2 + \pi_2^1\pi_1\right] +$$
$$\sigma^4 E\left[\pi_1^1\pi_3 + \pi_2^1\pi_2 + \pi_3^1\pi_1\right]$$

given the predictive risk of $\hat{T}b^*$ as stated (4.3) in theorem (4.1).

Theorem 4.2

Using (2.9) in the expression (2.11), we observe that

(6.9) $$\hat{T}b_S^* = X\beta + \sigma\theta_1 + \sigma^2\theta_2 + \sigma^3\theta_3 + \sigma^4\theta_4$$

Subtracting (6.5) from (6.9), we observe that

(6.10) $$\hat{T}b_s^* - E(Y) = \sigma\theta_1 + \sigma^2\theta_2 + \sigma^3\theta_3 + \sigma^4\theta_4$$

where

(6.10) $$\theta_1 = X\left(X'W^{-1}X\right)^{-1}X'W^{-1}u$$

(6.11) $$\theta_2 = -\left[\frac{T}{\alpha}X\left(X'W^{-1}X\right)^{-1}\beta + K\frac{u'Bu}{\beta'X'X\beta}X\beta\right]$$

(6.12) $$\theta_3 = -\left[\begin{array}{l}\frac{T}{\alpha}X'W^{-1}Xu\left(X'W^{-1}X\right)^{-2} + K \\ \frac{u'Bu}{\beta'X'X\beta}X\left(X'W^{-1}X\right)^{-1}X'W^{-1}u + \frac{2K}{\alpha}\beta'\end{array}\right.$$
$$X\left[\left(X'X\right)^{-1} - \left(X'W^{-1}X\right)^{-2}\right]\frac{TX'u\beta}{\beta'X'X\beta}$$
$$\left. - 2K\frac{u'Bu}{\beta'X'X\beta}\beta'X'u.X\beta\right]$$

(6.13) $$\theta_4 = \frac{1}{\beta'X'X\beta}\left[\begin{array}{l}K\frac{u'Bu}{\beta'X'X\beta}X\beta \\ \left(X'W^{-1}X.X'u - 2\frac{T}{\alpha}\beta'X'X\beta\right)\end{array}\right.$$
$$+\frac{2K}{\beta'X'X\beta}\left\{\beta'X'uX\left(\begin{array}{l}u'Bu\left(X'W^{-1}X\right)^{-1}X'W^{-1}u \\ +\frac{2}{\alpha}X\beta'\left[\begin{array}{l}\left(X'X\right)^{-1} \\ -\left(X'W^{-1}X\right)^{-2}\end{array}\right]TX'u\beta\end{array}\right)\right\}$$
$$+\frac{K}{\alpha}\left(X'W^{-1}X\right)^{-1}\left\{X\beta\left(\begin{array}{l}Tu'Bu - \frac{\beta'}{\alpha} \\ \left(X'W^{-1}X\right)^{-1}\left(X'X\right)^2\end{array}\right)\beta\right\}$$
$$\left. + 2K\beta'\left\{\left(X'X\right)^{-1} - \left(X'W^{-1}X\right)^{-2}\right\}TX'u'.XX'W^{-1}u\right]$$

The result of bias vector associated with theorem (4.2) can be obtained from

(6.14) $$E\left[\hat{T}b_s^* - E(Y)\right] = \sigma E(\theta_1) + \sigma^2 E(\theta_2)$$
$$+ \sigma^3 E(\theta_3) + \sigma^4 E(\theta_4)$$

where, $E[\theta_1] = 0$

Now, by taking expectation of

$$E\left[\theta_1{}'\theta_1\right];\ E\left[\theta_1{}'\theta_2\right];\ E\left[\theta_1{}'\theta_3\right]$$
$$E\left[\theta_2{}'\theta_1\right];\ E\left[\theta_2{}'\theta_2\right];\ E\left[\theta_3{}'\theta_1\right]$$

Utilizing these expectation in the expression

(6.15)
$$PR\left(\hat{T}b_s^*\right) = E\left[\left\{\hat{T}b_s^* - E(Y)\right\}' \left\{\hat{T}b_s^* - E(Y)\right\}\right]$$

$$= \sigma^2 E\left[\theta_1^1\theta_1\right] + \sigma^3 E\left[\theta_1^1\theta_2 + \theta_2{}'\theta_1\right]$$

$$+ \sigma^4 E\left[\theta_1^1\theta_3 + \theta_2{}'\theta_2 + \theta_3^1\theta_1\right]$$

given the predictive risk of $\hat{T}b_s^*$ as stated (4.5) in theorem (4.2)

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