3- Class Association Schemes from Hadamard matrix of Paley type I

Pandey, Pinky

Assistant Professor, Department of Mathematics, Nirmala College, Ranchi, 834002, Jharkhand, INDIA

DOI: 10.29322/IJSRP.13.06.2023.p13806
http://dx.doi.org/10.29322/IJSRP.13.06.2023.p13806

Paper Received Date: 15th April 2023
Paper Acceptance Date: 26th May 2023
Paper Publication Date: 6th June 2023

Abstract- In this paper we have constructed Association Schemes from Hadamard matrix of Paley type I. Paley type I Hadamard matrices are skew symmetric in nature. These Association Schemes so obtained are Amorphic. Association schemes.

***Index Terms*-** Hadamard matrices, Association Scheme, Skew Symmetric Hadamard matrix, Amorphic Association Scheme, Strongly Regular Graph, Paley's Hadamard matrix, Paley's Hadamard matrix of type I.

I. Introduction

We begin with the following definitions:

1.1 Hadamard Matrices (Or H-Matrices): Hadamard matrix is a square matrix whose entries are either +1 or −1 and whose rows are mutually orthogonal. If H-matrix of order n exists and n > 2 then n = 4t, where t is an integer. For a brief surveys of H-matrices vide Hall [1], Hedayat and Wallis [2]. For recent constructions vide Horadam [3]. Horton et. al. [4] and Baliga and Horadam [5].

1.2 Association Scheme (AS) vide Brouwer and Haemers [6], Colbourn and Dinitz [7]: Given v treatments 1,2,……,v, a relation satisfying the following conditions are said to be an Association Scheme with m classes: a) Any two treatments are either 1st, 2nd, ……….., or mth associates, the relation of association being symmetric. b) Each treatment has n_i, ith associates, the number n_i being independent of the treatment taken. c) If any two treatments α and β are ith associates, then the number of treatments which are jth associates of α and kth associates of β is p_{jk}^{i} and is independent of the pair of ith associates α and β. Let $R_0, R_1, \ldots, R_m$ be binary relations on a set X = {1,2,…., v}. Let $A_i = [a_{ij}]$ be the matrix with entries 0 and 1defined as $a_{jk} = \begin{cases} 1, \text{if } (j,k)\in R_i \\ 0, \text{otherwise} \end{cases}$. A_i is the adjacency matrix of R_i.

The set P = $(R_0, R_1, \ldots, R_m)$ is called an m–class association scheme if the following conditions are satisfied:

(i) $A_0 = I$ (Identity Matrix) and $A_i \neq 0, \forall i$

(ii) $\sum_{i=0}^{m} A_i = J$, where J is all-1 matrix

(iii) $A_i^T = A_i \ \forall$ i ε {0,1,2,……..,m}

(iv)There are numbers p_{ij}^k such that

$$A_i A_j = \sum_{k=0}^{m} p_{ij}^k A_k$$

1.3 Skew symmetric Hadamard matrix: A (1, −1) matrix A of order n is said to be of skew type if $A - I_n$ is skew-symmetric.

1.4 Amorphic Association Scheme: Let a 3-AS be defined by the association matrices I, A_1, A_2, A_3. Then 3-AS is called amorphic if each of A_1, A_2, A_3 is an adjacency matrix of a strongly regular graph.

1.5 Strongly Regular Graph (SRG) (vide Higman [8]): 2-associate Association scheme on a set X is also called Strongly Regular Graph (SRG). The parameters of SRG are (v, k, λ, μ) where v = order of association matrix, k = p_{11}^0, λ = p_{11}^1, μ = p_{11}^2.

1.6 Paley's method of construction for H-matrices: Paley [9] found two families of Hadamard matrices using the quadratic residues in a finite field GF (q), where q = p^n, where p is an odd prime. The quadratic character χ on the cyclic group GF $(q)^*$ = GF (q)-{0}, defined by

i) If g is a quadratic residue in GF (q) then χ (g) = 1 and
ii) if g is a quadratic non residue, χ (g) = -1

Also χ (0) = 0. For q, an odd prime power and an ordering { $g_0 = 0, g_1, ..., g_{q-1}$ } of GF (q), take Q = [χ (g_i - g_j)], $0 \le i, j < q$. Let S be a matrix of order (q + 1) x (q + 1). Take $S = \begin{bmatrix} 1 & e \\ e^T & Q \end{bmatrix}$ where e is a 1 X q array having all entries 1.

1.7 Hadamard matrix of Paley type I: H-matrices of Paley type I are defined for orders N = 4m = p+1 (m=1,2,3,...), where p is a prime with mod(p,4)=3. If *q* is congruent to 3 (mod 4) then

$$P_{q+1} = \begin{bmatrix} 1 & e \\ e^T & Q + I_q \end{bmatrix}$$

is a Hadamard matrix of order (q + 1).

Construction of Amorphic Association Schemes from Skew Symmetric Hadamard matrix of Paley type I:
Following theorem describes the construction of the Amorphic 3-AS

Theorem : If a Paley type I Hadamard matrix H is $\begin{bmatrix} 1 & e \\ e^T & I - \beta_1 + \beta_2 \end{bmatrix}$ where e is 1 X q array with all entries 1 and β_1, β_2 are (0,1) matrices, then

$$A_1 = \beta_1 \times \beta_2 + \beta_2 \times \beta_1$$

$$A_2 = \beta_1 \times \beta_1 + \beta_2 \times \beta_2$$

$$A_3 = I \times K + K \times I, \quad \text{where } K = \beta_1 + \beta_2$$

define an amorphic 3-AS .

Proof: When 4n-1 is equal to $p^r = q$, where p is a prime and let x is a primitive element of Galois field GF (p^r). Let { $1, x^2, (x^2)^2, (x^2)^3, ...(x^2)^{\frac{q-3}{2}}$ } mod (4n-1) is a difference set. We denote this difference set as { $1, d_1, d_2, ..., d_k$ } mod (4n-1) where $k = \frac{q-3}{2}$

Let $\alpha = \text{circ } (0100............0)$

Let $\beta_1 = \alpha + \alpha^{d_1} + \alpha^{d_2} + ... + \alpha^{d_k}$

and $\beta_2 = \alpha^{-1} + \alpha^{-d_1} + \alpha^{-d_2} + ... + \alpha^{-d_k}$

Then $\beta_1 \beta_2 = (k+1)\, I + (n-1)\, K$ where $4n - 1 = q$

$$= \left(\frac{q-1}{2}\right) I + \left(\frac{q-3}{4}\right) K \qquad (1)$$

Since $\beta_1 + \beta_2 = K \Rightarrow \beta_2 = K - \beta_1$

From (1) we have, $\beta_1 (K - \beta_1) = \left(\frac{q-1}{2}\right) I + \left(\frac{q-3}{4}\right) K$

$$\Rightarrow K\beta_1 - \beta_1^2 = \left(\frac{q-1}{2}\right) I + \left(\frac{q-3}{4}\right) K \qquad (2)$$

ISSN 2250-3153

As β_1 is regular $(0,1)$ matrix, $\beta_1 J = J\beta_1 = \left(\frac{q-1}{2}\right)J$ and $\beta_2 J = J\beta_2 = \left(\frac{q-1}{2}\right)J$

Also, $\beta_1 K = \beta_1(J - I) = \beta_1 J - \beta_1$

$$= \left(\frac{q-1}{2}\right)J - \beta_1$$

$$= \left(\frac{q-1}{2}\right)(I + K) - \beta_1$$

From (2), we have

$$\left(\frac{q-1}{2}\right)I + \left(\frac{q-1}{2}\right)K - \beta_1 - \beta_1^2 = \left(\frac{q-1}{2}\right)I + \left(\frac{q-3}{4}\right)K$$

$$\Rightarrow \beta_1^2 = \left(\frac{q-1}{2}\right)K - \left(\frac{q-3}{4}\right)K - \beta_1$$

$$= \left(\frac{q+1}{4}\right)K - \beta_1$$

$$= \left(\frac{q+1}{4}\right)(\beta_1 + \beta_2) - \beta_1$$

$$= \left(\frac{q-3}{4}\right)\beta_1 + \left(\frac{q+1}{4}\right)\beta_2$$

Similarly, $\beta_2^2 = \left(\frac{q+1}{4}\right)\beta_1 + \left(\frac{q-3}{4}\right)\beta_2$

If $t = \left(\frac{q-3}{4}\right) \Rightarrow \left(\frac{q+1}{4}\right) = (t+1)$ and $\left(\frac{q-1}{2}\right) = (2t+1)$

$$\therefore \beta_1^2 = t\,\beta_1 + (t+1)\beta_2$$

$$\therefore \beta_2^2 = (t+1)\,\beta_1 + t\beta_2$$

$$\therefore \beta_1\,\beta_2 = (2t+1)I + tK$$

Let $A_1 = \beta_1 \times \beta_2 + \beta_2 \times \beta_1$

$A_2 = \beta_1 \times \beta_1 + \beta_2 \times \beta_2$

$A_3 = I \times K + K \times I$

Then $A_1^2 = \beta_1^2 \times \beta_2^2 + \beta_2^2 \times \beta_1^2 + 2(\beta_1\beta_2 \times \beta_2\beta_1)$

ISSN 2250-3153

$$= \{t\beta_1 + (t+1)\beta_2\} \times \{(t+1)\beta_1 + t\beta_2\} + \{(t+1)\beta_1 + t\beta_2\} \times \{t\beta_1 + (t+1)\beta_2\}$$

$$+ 2[\{(2t+1)I + tK\} \times \{(2t+1)I + tK\}]$$

$$= t(t+1)(\beta_1 \times \beta_1) + t^2(\beta_1 \times \beta_2) + (t+1)^2(\beta_2 \times \beta_1) + t(t+1)(\beta_2 \times \beta_2)$$

$$+ t(t+1)(\beta_1 \times \beta_1) + t^2(\beta_2 \times \beta_1) + (t+1)^2(\beta_1 \times \beta_2) + t(t+1)(\beta_2 \times \beta_2)$$

$$+ 2[(2t+1)^2 I + (2t+1)\,t\,(I \times K) + (2t+1)\,t\,(K \times I) + t^2(K \times K)]$$

$$= 2t(t+1)A_2 + t^2 A_1 + (t+1)^2 A_1 + 2(2t+1)^2 I + 2t(2t+1)A_3 + 2t^2(A_1 + A_2)$$

$$= 2(2t+1)^2 I + (4t^2 + 2t + 1)A_1 + 2t(2t+1)(J - A_1 - I)$$

And $A_2^2 = \beta_1^2 \times \beta_1^2 + \beta_2^2 \times \beta_2^2 + 2(\beta_1\beta_2 \times \beta_2\beta_1)$

$$= \{t\beta_1 + (t+1)\beta_2\} \times \{t\beta_1 + (t+1)\beta_2\} + \{(t+1)\beta_1 + t\beta_2\} \times \{(t+1)\beta_1 + t\beta_2\}$$

$$+ 2[\{(2t+1)I + tK\} \times \{(2t+1)I + tK\}]$$

$$= t(t+1)(\beta_1 \times \beta_2) + t^2(\beta_1 \times \beta_1) + (t+1)^2(\beta_2 \times \beta_2) + t(t+1)\beta_2 \times \beta_1$$

$$+ t(t+1)(\beta_1 \times \beta_2) + t^2(\beta_2 \times \beta_2) + (t+1)^2(\beta_1 \times \beta_1) + t(t+1)\beta_2 \times \beta_1$$

$$+ 2[(2t+1)^2 I + (2t+1)\,t\,(I \times K) + (2t+1)\,t\,(K \times I) + t^2(K \times K)]$$

$$= (2t^2 + 2t + 1)A_2 + 2t(t+1)A_1 + (t+1)^2 A_1 + 2(2t+1)^2 I + 2t(2t+1)A_3 + 2t^2(A_1 + A_2)$$

$$= 2(2t+1)^2 I + (4t^2 + 2t + 1)A_2 + 2t(2t+1)(J - A_2 - I)$$

And $A_3^2 = I \times K^2 + K^2 \times I + 2(K \times K)$

$$= I \times \{(4t+2)I + (4t+1)K\} + \{(4t+2)I + (4t+1)K\} \times I + 2(A_1 + A_2)$$

$$= (4t+2)I + (4t+1)(I \times K) + (4t+2)I + (4t+1)(K \times I) + 2(A_1 + A_2)$$

$$= 4(2t+1)I + (4t+1)A_3 + 2(J - A_3 - I)$$

ISSN 2250-3153

And $A_1A_2 = \beta_1^2 \times \beta_2\beta_1 + \beta_1\beta_2 \times \beta_2^2 + \beta_2\beta_1 \times \beta_1^2 + \beta_2^2 \times \beta_1\beta_2$

$$= \{(t\beta_1 + (t+1)\beta_2\} \times \{(2t+1)I + tK\} + \{(2t+1)I + tK\} \times \{(t+1)\beta_1 + t\beta_2\}$$

$$+ \{(2t+1)I + tK\} \times \{(t\beta_1 + (t+1)\beta_2\} + \{(t+1)\beta_1 + t\beta_2\} \times \{(2t+1)I + tK\}$$

$$= (2t+1)^2\{K \times I + I \times K\} + (4t^2 + 2t)\{K \times K\}$$

$$= (2t+1)^2 A_3 + (4t^2 + 2t)(A_1 + A_2)$$

And $A_1A_3 = \gamma_1 \times K\gamma_2 + K\gamma_1 \times \gamma_2 + \gamma_2 \times K\gamma_1 + K\gamma_2 \times \gamma_1$

$$= \gamma_1 \times \{(2t+1)I + (2t+1)K - \gamma_2\} + \{(2t+1)I + (2t+1)K - \gamma_1\} \times \gamma_2 +$$

$$\gamma_2 \times \{(2t+1)I + (2t+1)K - \gamma_1\} + \{(2t+1)I + (2t+1)K - \gamma_2\} \times \gamma_1$$

$$= (2t+1)\gamma_1 \times I + (2t+1)\gamma_1 \times K - \gamma_1 \times \gamma_2 + (2t+1)I \times \gamma_2 + (2t+1)K \times \gamma_2 - \gamma_1 \times \gamma_2 +$$

$$(2t+1)\gamma_2 \times I + (2t+1)\gamma_2 \times K - \gamma_2 \times \gamma_1 + (2t+1)I \times \gamma_1 + (2t+1)K \times \gamma_1 - \gamma_2 \times \gamma_1$$

$$= 4tA_1 + (4t+2)A_2 + (2t+1)A_3$$

And $A_2A_3 = \gamma_1 \times \gamma_1 K + K\gamma_1 \times \gamma_1 + \gamma_2 \times \gamma_2 K + K\gamma_2 \times \gamma_2$

$$= \gamma_1 \times \{(2t+1)I + (2t+1)K - \gamma_1\} + \{(2t+1)I + (2t+1)K - \gamma_1\} \times \gamma_1 +$$

$$\gamma_2 \times \{(2t+1)I + (2t+1)K - \gamma_2\} + \{(2t+1)I + (2t+1)K - \gamma_2\} \times \gamma_2$$

$$= (2t+1)\gamma_1 \times I + (2t+1)\gamma_1 \times K - \gamma_1 \times \gamma_1 + (2t+1)I \times \gamma_1 + (2t+1)K \times \gamma_1 - \gamma_1 \times \gamma_1 +$$

$$(2t+1)\gamma_2 \times I + (2t+1)\gamma_2 \times K - \gamma_2 \times \gamma_2 + (2t+1)I \times \gamma_2 + (2t+1)K \times \gamma_2 - \gamma_2 \times \gamma_2$$

$$= (4t+2)A_1 + 4tA_2 + (2t+1)A_3$$

Hence A_1, A_2 and A_3 define an amorphic 3 - AS.

Conclusion: Association schemes from skew-symmetric Hadamard matrices have also been obtained by Hanaki [10]. However our method is different from that of Hanaki.

REFERENCES

[1] 1. M. Hall Jr., Combinatorial Theory,Wiley, New York 2nd edition, 1986.
[2] 2. A. Hedayat and W. D. Wallis, Hadamard matrices and their applications, Ann. Statist., 6 (1978) 1184-1238.
[3] 3. K.J. Horadam, Hadamard Matrices and Their Applications, Princeton University Press (2007).
[4] 4. J. Hortan, C. Koukouvinos and J. Seberry, A Search for Hadamard Matrices Constructed from Williamson Matrices, Bull, ICA 35 (2002) 75-88 http://www.uow.edu.au/~jennie/hadamard.html.
[5] 5. A. Baliga and K. J. Horadam, Cocyclic Hadamard Matrices over Australas J. Combin, 11 (1995) 123-134.
[6] 6. A.E. Brouwer & W.H. Haemers, Association schemes, 1987.
[7] 7. C. J. Colbourn, and J. H. Dinitz. The CRC Handbook of Combinatorial designs, CRC Press, Second Edition (2007), ISBN- 9780367248291.
[8] 8. D.G. Higman, Strongly Regular Graph and Coherent Configurations of type[ ],European J. of Combinatorics, 9 (1988) 411-422.
[9] 9. R.E.A.C. Paley, On Orthogonal matrices, J. Math. Phy. 12 (1933) 311-320.
[10] 10.A. Hanaki, Skew-Symmetric Hadamard Matrices and Association Scheme, SUT Journal of Mathematics Vol 36 No. 2 (2000), 251-258.

AUTHORS

First Author – Pandey, Pinky, Assistant Professor, Department of Mathematics, Nirmala College, Ranchi, 834002, Jharkhand, INDIA
Email Id – unix.pinky@gmail.com