ISSN 2250-3153

Some Notes on the Solution of Sturm-Liouville Boundary-Value Problems Having Polynomial Coefficients using Laplace Transforms

Roberto P. Briones
School of Mathematical and Computer Sciences
Heriot-Watt University
Putrajaya, Malaysia

DOI: 10.29322/IJSRP.11.06.2021.p11425
http://dx.doi.org/10.29322/IJSRP.11.06.2021.p11425

Abstract – This paper introduces a procedure to apply Laplace transform theory in solving a class of Sturm – Liouville boundary – value problems, namely those having polynomial coefficients. The method demonstrates that not only can Laplace transforms solve differential equations satisfying only initial-value conditions, but can also be adaptable to solving boundary – value problems. In the end, the application of inverse Laplace transforms to the problem introduces the reader to a novel approach in the analysis and solution of a specific class of Sturm – Liouville boundary problems.

Index Terms- Sturm – Liouville boundary – value problems, Laplace transforms, Inverse Laplace transforms, convolution of functions, Laurent series expansion

1. INTRODUCTION

The Sturm-Liouville boundary problem is defined as follows [1]:

$$Ly(t) = -\lambda r(t)y(t), \text{ where } Ly = \left(p(t)y'(t)\right)' + q(t)y(t), \text{ and}$$

$$\begin{cases} \alpha_1 y(0) + \alpha_2 y'(0) = 0 & (1) \\ \beta_1 y(l) + \beta_2 y'(l) = 0 & (2) \end{cases},$$

$$\text{where } \; p(t) > 0, r(t) > 0.$$

We now discuss the solution to the SLBVP when the $p(t), r(t)$, and $q(t)$ are polynomials over $[0, l]$ and $p(t), r(t) \; > \; 0.$

The Laplace transform of a function f(t) is defined as

$$\mathcal{L}\{f(t)\} \equiv F(s) = \int_0^\infty e^{-st} f(t)dt,$$

whenever the improper integral exists. From here on out we assume that the function $f(t)$ is piecewise continuous over the interval $[0, \infty)$, and that it is of exponential order α. Recall that a function $f(t)$ is of exponential order α for $t > T$ if there are constants M and α such that $|f(t)| \leq Me^{\alpha t}$ for $t > T$. If $f(t)$ is piecewise continuous in every finite interval $[0, T]$ and of exponential order α for $t > T$, then we are assured that the Laplace transform of $f(t)$ exists for $s > \alpha$ [2].

In this paper we set out to apply the tools of Laplace transform theory to solving a selected class of Sturm – Liouville boundary value problems, namely those having *polynomial* coefficients. Laplace transforms are applicable to solving differential equations satisfying initial boundary conditions. After minor adjustments it can be shown to be able to solve linear boundary value problems with constant coefficients [3]. The Laplace transform was also used to determine an expression for the temperature for the heat transfer of incompressible viscous nanofluids through carbon nanotubes, which is guided by a nonlinear second – order boundary value problem [4].

2. PRELIMINARIES

Lemma 2.1. Let $f(t)$ be a piecewise continuous function over $[0, l]$, differentiable over $(0, l)$, and of exponential order over $[0, \infty)$. Then

a) $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$, and
b) $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$
c) $\mathcal{L}\left\{f^{(n)}(t)\right\} = s^nF(s) - sf^{(n-1)}(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$

Proof. The proofs of these statements are straightforward and the reader is referred to any standard text on the theory of Laplace transforms for their proofs. In particular, the reader is referred to Sanchez et al [5]. ■

Lemma 2.2. $\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$.

Lemma 2.3. If $\mathcal{L}\{f(t)\} = F(s)$ then $\frac{d^n}{ds^n}\mathcal{L}\{f'(t)\} = nF^{(n-1)}(s) + sF^{(n)}(s)$ for $n \geq 1$.

Proof. This can be shown inductively. Referring to Lemma 2.1(a), we have for $n = 1$,

$\frac{d}{ds}\mathcal{L}\{f'(t)\} = F(s) + sF'(s)$. Assuming that the statement is true for n = k, we then have $\frac{d^k}{ds^k}\mathcal{L}\{f'(t)\} = kF^{(k-1)}(s) + sF^{(k)}(s)$. Then for $n = k + 1$, differentiating this last equation with respect to s gives

$\frac{d^{k+1}}{ds^{k+1}}\mathcal{L}\{f'(t)\} = kF^{(k)}(s) + F^{(k)}(s) + sF^{(k+1)}(s) = (k+1)F^{(k)}(s) + sF^{(k+1)}(s)$, and the statement is proved. ■

Theorem 2.4. $\mathcal{L}\{t^n f'(t)\} = (-1)^n\{nF^{(n-1)}(s) + sF^{(n)}(s)\}$.

Proof. This statement follows directly from Lemmas 2.2 and 2.3. ■

Theorem 2.5. $\mathcal{L}\left\{\frac{d}{dt}(t^n f'(t))\right\} = \mathcal{L}\{(t^n f'(t))'\} = (-1)^n s\{nF^{(n-1)}(s) + sF^{(n)}(s)\}$.

Proof. This statement follows directly by applying Lemma 2.1(a) to Theorem 2.4.■

Theorem 2.5 is the key to applying the Laplace transform to the solution of Sturm – Liouville boundary value problems. Note that any solution $y(t)$ to the SLBVP already incorporates the initial conditions at $t = 0$ into Theorem 2.5, and enables us to focus only at the boundary conditions at $t = l$ (which entails solving for the eigenvalues of the SLBVP) at the end of the solution of the problem.

3. EXAMPLES

Example 3.1. Solve the Sturm – Liouville problem with the following Robin boundary conditions:

$$\left(t^2 f'(t)\right)' = -\lambda t^2 f(t)$$

$$f'(0) = \alpha_1 f(0),\ \ f'(l) = \alpha_2 f(l)$$

Solution.

Case 1. $\lambda = 0$.

Applying Theorem 2.5 to the Sturm-Liouville equation one obtains

$2sF'(s) + s^2F''(s) = 0$.

$\Rightarrow 2F'(s) + sF''(s) = 0$, since $s > 0$.

$\Rightarrow \ln F'(s) = -2\ln s + \ln k$, or $\ln F'(s) = \ln\left(\frac{1}{s^2}\right) + \ln k = \ln\frac{k}{s^2} \Rightarrow F'(s) = \frac{k}{s^2}$.

$\Rightarrow F(s) = -\frac{k}{s} + c \Rightarrow f(t) = -k + c\delta(t)$.

From this we get $f(0) = -k + c$, $f(l) = -k$, and $f'(l) = 0$. But since the derivative of the delta function could be $\pm\infty$ at t = 0, the only way $f(t)$ can possibly satisfy boundary condition (2) of the SLBVP is for $c = 0$. This implies that $f(t) = -k$. Boundary condition (1) then implies that $\alpha_1 f(0) = 0$, which results in $f(0) = 0$. If we want $f(t)$ to be continuous from the right at $t = 0$, the last statement would result in $f(t) = 0$, the trivial solution. Hence, we conclude that $\lambda = 0$ is not an eigenvalue of the SLBVP. (And the trivial solution is the only solution to the SLBVP.)

Case 2. $\lambda < 0$.

Let $\lambda = -\mu^2$. Applying Theorem 2.5 to the Sturm-Liouville equation one obtains

$2sF'(s) + s^2F''(s) = -\lambda F''(s)$

$\Rightarrow (s^2 - \mu^2)F''(s) = -2sF'(s)$

$\Rightarrow \frac{F''(s)}{F'(s)} = \frac{-2s}{s^2-\mu^2} = \frac{-2s}{(s-\mu)(s+\mu)} = -\frac{1}{s+\mu} - \frac{1}{s-\mu}$.

Integrating both sides with respect to s, one obtains

$\Rightarrow \ln F'(s) = -\ln(s+\mu) - \ln(s-\mu) - \ln\ k = -\ln\{k(s^2-\mu^2)\}$.

$\Rightarrow\ F'(s) = \frac{c}{s^2-\mu^2} = \frac{c}{2\mu}\left\{\frac{1}{s-\mu} - \frac{1}{s+\mu}\right\}$

$\Rightarrow\ F(s) = \frac{c}{2\mu}\ln\left(d\left(\frac{s-\mu}{s+\mu}\right)\right)$, for some constants c and d.

$\Rightarrow\ f(t) = \frac{c}{2\mu}\cdot\left(\frac{e^{-\mu t}-e^{\mu t}}{t} + \ln d\cdot\delta(t)\right)$, or $f(t) = \frac{c}{2\mu}\cdot\left(\ln d\cdot\delta(t) - \frac{2\sinh(\mu t)}{t}\right)$.

The inverse Laplace transform was obtained by use of Wolfram Alpha (https://www.wolframalpha.com/), considering the uncommon form of $F(s)$.

To compute for d, firstly after substituting $t = 0$ into $f(t)$, we get

$$f(0) = \frac{c}{2\mu}\left((\ln d)\delta(0) - 2\mu\right)$$

From the expression for $f(t)$ we obtain

$\Rightarrow\ f'(t) = \frac{c}{2\mu}\left((\ln d)\delta'(t) - 2\frac{\mu\, t\cosh(\mu t)-\sinh(\mu t)}{t^2}\right)$

$\Rightarrow\ f'(0) = \frac{c}{2\mu}(\ln d)\delta'(0)$.

The first Robin condition $f'(0) = \alpha_1 f(0)$ then implies that $\ln d = \frac{2\mu\alpha_1}{c_1\delta(0)-\delta'(0)}$, or $d = e^{\frac{2\mu\alpha_1}{c_1\delta(0)-\delta'(0)}}$. Replacing this expression into $f(0)$ above yields $c = \frac{f(0)}{\delta'(0)}(c_1\delta(0) - \delta'(0))$.

From the boundary condition $f'(l) = \alpha_2 f(l)$, a bit of algebra gives the equation

$$\frac{\mu l}{\alpha_2 l+1} = \tanh(\mu l),$$

which becomes the source of the eigenvalues of the SLBVP when $\lambda = -\mu^2 < 0$.

Case 3. $\lambda > 0$.

Let $\lambda = \mu^2$. Applying Theorem 1.5 to the Sturm-Liouville equation one obtains

$2sF'(s) + s^2F''(s) = -\lambda F''(s) \Rightarrow (s^2+\mu^2)F''(s) = -2sF'(s)$

$\Rightarrow \frac{F''(s)}{F'(s)} = \frac{-2s}{s^2+\mu^2} \Rightarrow \ln F'(s) = \ln\left(\frac{c}{s^2+\mu^2}\right)$, after integrating both sides with respect to s.

$\Rightarrow\ F'(s) = \frac{c}{s^2+\mu^2}$

$\Rightarrow F(s) = \frac{c}{\mu}\left(tan^{-1}\left(\frac{s}{\mu}\right) + d\right)$, for some constants c and d.

With the aid of Wolfram Alpha, it is found that

$$f(t) = \frac{c}{\mu}\cdot\left(\left(\frac{\pi}{2} + d\right)\delta(t) - \frac{\sin(\mu t)}{t}\right).$$

A similar computation to Case 2 yields the value $d = \frac{\alpha_2\mu}{\alpha_2\delta(0)-\delta'(0)} - \frac{\pi}{2}$.

Using this value into the function $f(t)$ above, and making use of the second Robin condition yields the equations $c = \frac{\alpha_1 f(0)(\alpha_2\delta(0)-\delta'(0))}{\alpha_2\delta'(0)}$ and $\tan(\mu l) = \frac{\mu l}{\alpha_2 l+1}$, whence come the eigenvalues of the SLBVP.

If we let the function $f(t)$ to be continuous from the right at $t = 0$, the solution then becomes the function

$$f(t) = \frac{c}{\mu} \cdot \frac{\sin(\mu t)}{t}$$

The last function results in $c = f(0)$, so that $f(t) = f(0)\frac{\sin(\mu t)}{\mu t}$. ■

If we assume that $f(t)$ is continuous from the right at $t = 0$, then it becomes clear that

$f(t) = \frac{c}{2\mu} \cdot \left(\ln d \cdot \delta(t) - \frac{2\sinh(\mu t)}{t}\right)$ becomes $f(t) = -\frac{c}{\mu} \cdot \frac{\sinh(\mu t)}{t} = f(0)\frac{\sinh(\mu t)}{\mu t}$

In Example 3.1 we were a bit fortunate that the simplicity of the structure of the SLBVP enabled us to solve the problem without much issue. In general, given a SLBVP with arbitrary polynomial coefficients is very hard to solve, even with the use of sophisticated computing technology. An addition of an extra term could change the texture of a prior approach to a solution in a different way, as we shall see below. In the next example, by virtue of computational complexity, we will be content at arriving at the general form of $f(t)$ only, and shall skip the actual calculation of the eigenvalues.

Example 3.2. Solve the Sturm – Liouville boundary problem with the following boundary conditions:

$$\left(t^2 f'(t)\right)' + tf(t) = -\lambda t^2 f(t)$$

$$\begin{cases} \alpha_1 f(0) + \alpha_2 f'(0) = 0 & (1) \\ \beta_1 f(l) + \beta_2 f'(l) = 0 & (2) \end{cases}$$

Case 1. λ = 0.

An application of Theorem 2.5 to the differential equation gives

$$2sF'(s) + s^2 F''(s) - F'(s) = 0$$

$\Rightarrow s^2 F''(s) = (-2s + 1)F'(s) \Rightarrow \frac{F''(s)}{F'(s)} = -\frac{2}{s} + \frac{1}{s^2} \Rightarrow \ln F'(s) = -2\ln s - \frac{1}{s} + \ln c.$

$\Rightarrow \ln F'(s) = \ln\left(\frac{c}{s^2 e^{1/s}}\right) \Rightarrow F'(s) = \frac{c}{s^2 e^{1/s}} \Rightarrow F(s) = ce^{-\frac{1}{s}} + d.$

Using Wolfram Alpha to evaluate the inverse Laplace transform, one obtains

$f(t) = c\left(\delta(t) - \frac{J_1(2\sqrt{t})}{\sqrt{t}}\right) + d\delta(t).$

Assuming right-hand continuity for our function $f(t)$ at $t = 0$ will thus produce the solution

$f(t) = -c\frac{J_1(2\sqrt{t})}{\sqrt{t}}$. Calculation of the value of the coefficient c will then proceed from the boundary condition $f'(l) = c_2 f(l)$.

Case 2. $\lambda < 0$. (Let $\lambda = -\mu^2$).

The differential equation then becomes

$$2sF'(s) + s^2 F''(s) - F'(s) = -\lambda F''(s).$$

$$\Rightarrow \frac{F''(s)}{F'(s)} = -\frac{2s - 1}{s^2 - \mu^2} = \frac{\frac{1}{2\mu} - 1}{s - \mu} - \frac{\frac{1}{2\mu} + 1}{s + \mu}$$

$$\Rightarrow \ln F'(s) = \left(\frac{1}{2\mu} - 1\right)\ln(s - \mu) - \left(\frac{1}{2\mu} + 1\right)\ln(s + \mu) + \ln c$$

$$\Rightarrow F'(s) = c\frac{(s - \mu)^{\frac{1}{2\mu} - 1}}{(s + \mu)^{\frac{1}{2\mu} + 1}}$$

$$\Rightarrow F(s) = c\left(\frac{s - \mu}{s + \mu}\right)^{\frac{1}{2\mu}} + d = c(s - \mu)^{\frac{1}{2\mu}} \cdot \frac{1}{(s + \mu)^{1/2\mu}} + d$$

$$\Rightarrow \; f(t) = c\int_0^t \frac{e^{\mu\tau}\tau^{-\frac{1}{2\mu}-1}}{\Gamma\left(-\frac{1}{2\mu}\right)} \cdot \frac{e^{-\mu(t-\tau)}(t-\tau)^{\frac{1}{2\mu}-1}}{\Gamma\left(\frac{1}{2\mu}\right)} d\tau + d\delta(t)$$

$$= \frac{ce^{-\mu t}}{\Gamma\left(-\frac{1}{2\mu}\right)\Gamma\left(\frac{1}{2\mu}\right)}\int_0^t e^{2\mu\tau}\tau^{-\frac{1}{2\mu}-1}(t-\tau)^{\frac{1}{2\mu}-1}d\tau + \; d\delta(t)$$

after applying the inverse Laplace transform of the Laplace transform of the convolution of two functions. Hence,

$$f(t) = \frac{ce^{-\mu t}}{\Gamma\left(-\frac{1}{2\mu}\right)\Gamma\left(\frac{1}{2\mu}\right)}\int_0^t e^{2\mu\tau}\tau^{-\frac{1}{2\mu}-1}(t-\tau)^{\frac{1}{2\mu}-1}d\tau + \; d\delta(t).$$

The integral above does not have a closed – form expression, but if we allow the Laurent series expansion for $F(s)$, then one obtains through Wolfram Alpha,

$$\Rightarrow \; F(s) = c\left(\frac{s-\mu}{s+\mu}\right)^{\frac{1}{2\mu}} + d = c\left(1 - \frac{1}{s} + \frac{1}{2s^2} + \frac{-2\mu^2-1}{6s^3} + O\left(\frac{1}{s^4}\right)\right) + d$$

$$\Rightarrow f(t) = \; c\left(\delta(t) - 1 + \frac{t}{2} - \frac{2\mu^2+1}{12}t^2 + O(t^3)\right) + d\delta(t).$$

A series expansion of $f(t)$, along with the boundary conditions, would theoretically enable us to produce approximate values of the eigenvalues of the SLBVP.

Case 3. $\lambda > 0$. (Let $\lambda = \mu^2$).

Similar to Case 2 one is led to the same differential equation $2sF'(s) + s^2F''(s) - F'(s) = -\lambda F''(s)$, but for this case the following arrangement is obtained:

$\Rightarrow \; \frac{F''(s)}{F'(s)} = -\frac{2s-1}{s^2+\mu^2} = \frac{-2s+1}{s^2+\mu^2}$. This gives

$\ln F'(s) = -\ln(s^2+\mu^2) + \frac{1}{\mu}tan^{-1}\left(\frac{s}{\mu}\right) + \ln c,$

after integration. The equation then simplifies as follows:

$$\ln F'(s) = \ln\left(c\frac{e^{\frac{1}{\mu}tan^{-1}\left(\frac{s}{\mu}\right)}}{s^2+\mu^2}\right) \Rightarrow \; F'(s) = c\frac{e^{\frac{1}{\mu}tan^{-1}\left(\frac{s}{\mu}\right)}}{s^2+\mu^2}$$

$\Rightarrow F(s) = \; ce^{\frac{1}{\mu}tan^{-1}\left(\frac{s}{\mu}\right)} + d$ after another integration.

The inverse Laplace transform for this last equation has a result that practically cannot be expressed in standard mathematical functions even with the help of Wolfram Alpha. However, just like in the previous case, if we consider the Laurent expansion for $F(s)$, then one obtains

$$\Rightarrow F(s) = \; ce^{\frac{1}{\mu}tan^{-1}\left(\frac{s}{\mu}\right)} + d = ce^{1/2\mu\pi}\left(1 - \frac{1}{s} + \frac{1}{2s^2} + \frac{2\mu^2-1}{6s^3} + O\left(\frac{1}{s^4}\right)\right) + d$$

$$\Rightarrow f(t) = \; ce^{1/2\mu\pi}\left(\delta(t) - 1 + \frac{t}{2} + \frac{2\mu^2-1}{12}t^2 + O(t^3)\right) + d\delta(t)$$

And similar to the previous case, a series expansion of $f(t)$, along with the boundary conditions, would theoretically enable us to produce approximate values of the eigenvalues of the SLBVP.

4. CONCLUSION

In this paper we have demonstrated how the theory of Laplace transforms can be used to analyse and solve some class of Sturm – Liouville boundary value problems, namely those which have polynomial coefficients. In general, it is difficult to solve SLBVP with arbitrary functional coefficients, and much effort has been spent on solving such differential equations with constant coefficients. This paper was able to generalize some results in the theory of Laplace transforms and make them adaptable to the solution of SLBVP's. The examples provided in this paper serve not only to present the procedure in using Laplace transform theory to solve SLBVPS's but also how it can serve as a springboard for its general analysis.

5. REFERENCES

1. R. L. Herman, *Second Course in Ordinary Differential Equations: Dynamical Systems and Boundary Value Problems*. ODE Notes (uncw.edu), 2008, ch. 6
2. M. R. Spiegel, *Schaum's Outline of Theory and Problems of Advanced Mathematics for Engineers and Scientists.* McGraw – Hill Book Co., 1971, ch. 4.
3. T. Angell, "Simple Two – Point Boundary Value Problems," http://www.math.udel.edu/~angell/BVP.pdf , undated.
4. H. Saleh, E. Alali, and A. Ebaid, "Medical applications for the flow of carbon – nanotubes suspended nanofluids in the presence of convective condition using Laplace transform," *Journal of the Association of Arab Universities for Basic and Applied Sciences*, 24:1, 206 – 212, DOI: 10.1016/j.jaubas.2016.12.001
5. D. A. Sanchez, R.C. Allen, and W.T Kyner, *Differential Equations.* Addison – Wesley Publishing Co, 1983, pp. 182 – 183.