A General Study on Schwarz-Christoffel Transformation and Its Applications

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Abstract: Two German Mathematician H.A.Schwarz (1843-1921) and E.B.Christoffel (1829-1900) established Schwarz-Christoffel transformation independently. The Schwarz-Christoffel transformation referred by many researchers due to have a vast applications in solving 2nd order partial differential equations like as Laplace's and Poisson's equations[3,4] and to study the different phenomena's of fluid flow. In this paper we would like to show the applications of Schwarz-Christoffel transformation directly to the solutions of problems in fluid flow and electrostatic potential theory.

Index Terms:Transformation, Polygon, Half plane, Analytic function, Degenerate Polygon, and Fluid flow.

Introduction

The Schwarz-Christoffel transformation is very much familiar as which maps the x-axis and the upper half of the z-plane onto a given simple closed polygon and its interior in the w-plane. Generally we represent the unit vector which is tangent to a smooth arc c at a point $z_0$by the complex number t and let the number τ denote the unit vector tangent to the image Γ of C at the corresponding point w_0 under a transformation w=f (z) .We assume that f is analytic at z_0 and that $f'(z_0) \neq 0$ .From the general rules of conformal mapping (preservation of angles)

We can write,arg τ =argf′(z_0)+arg t **(1)**

Particularly, if C is a segment of the axis with positive sense to right then t =1 and arg t = 0

At each point z_0= x on C, in that case equation (1) becomes arg τ =argf′(x) **(2)**

If f′(z) has a constant argument along the segment ,it follows that arg τ is constant . Hence the image Γ of C is also the segment of straight line.

Let us construct a transformation w=f(z) that maps the whole x axis onto a polygon of n sides where $x_1,x_2,x_3,\ldots..x_{n-1}$ and ∞ are the points on that axis whose images are to be the vertices of the polygon and where $x_1<x_2<x_3<\ldots..<x_{n-1}$

The vertices are the n points $w_j = f(x_j)$ where j= 1,2,3,…..n-1 and $w_n= f(\infty)$. The function f must be as such that argf′(z) jumps from one constant value to another at points z= x_j as the point z traces out of the x axis.

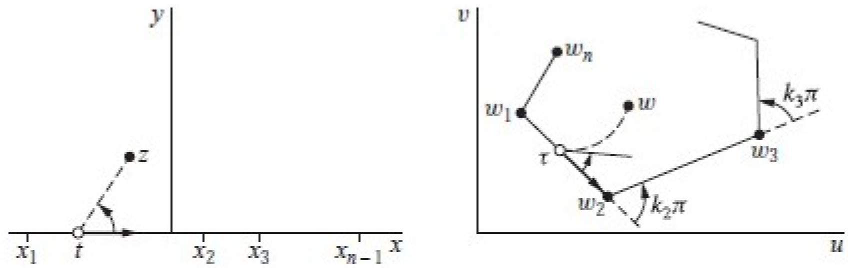

Figure: 01

If the function f is chosen or expressed such that

f′(z)=$A(z-x_1)^{-k_1}(z-x_2)^{-k_2}(z-x_3)^{-k_3}\ldots\ldots.(z-x_{n-1})^{-k_{n-1}}$**(3)**

Where A is the complex constant and each k_3 is a real constant then the argument of f′(z) changes in the prescribed manner as Z depicts the real axis then we can express the argument of the derivative (3) as

Arg f′ (z) = $ArgA - k_1Arg(z-x_1) \quad - k_2Arg(z-x_2) \quad \ldots\ldots - k_{n-1}Arg(z-x_{n-1})$ **(4)**

When z =x and x <x_1 then

Arg (z-x_1) =Arg (z-x_2) =-------= Arg (z-x_{n-1}) = π

When x_1<x<$x_2$theargument arg (z-x_1) is 0 and each of other argument is π.

According to the equation (4), thenArg f′(z) increases suddenly by the angle $k_1\pi$ as z moves to the right through the point z = x_1. It again jumps in value by the amount $k_2\,\pi$ as z passes through the point x_2. When a finite point z=x_n on the x axis instead of the point at infinity represents the point whose image is the vertex w_n.It exposes that the Schwartz-Christoffeltransformation takes of the form w= $A\int_{z_0}^{z}(s-x_1)^{-k_1}(s-x_2)^{-k_2}\ldots\ldots(s-x_n)^{-k_n}$ ds +B **(5)**

Where $k_1+k_2+k_3+\ldots..+k_n=2$ [1]

Research Elaborations

In this section wewill discussed Schwarz-Christoffel transformation to some degenerate polygons for which the integrals represent elementary functions. For purposes of research and materials here we discuss two examples in transformation where the Schwarz-Christoffel transformation maps the half plane onto the strip and the strip onto the half plane [2].

Example 1Here we show that the transformation w= sinz is a one one mapping of the semi infinite strip $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$; $y \geq 0$ in the z plane of the upper half v≥ 0 of the w plane.

At first we show that the boudary of the strip is mapped in a one one manner onto the real axis of the w plane as focused in figure -2.The image of the line segment BA there is found by writing x=$\frac{\pi}{2}$ in the equation u= sinx coshy and v= cosx sinhy and restricting y to be nonnegative.

Since u =coshy and v=0 when x = $\frac{\pi}{2}$,a typical point $(\frac{\pi}{2}, y)$ *on* BA is mapped onto the point (coshy,0) in the w plane and that image must move to ther right from B′ along the u axis as $(\frac{\pi}{2}, y)$ moves upward from B. A point (x,0)on the horizontal system DB has image (sinx,0)which moves to the right from D′ to B′ as x increases from x=-$\frac{\pi}{2}$ $to\ x = \frac{\pi}{2}$ or x goes from D to B.

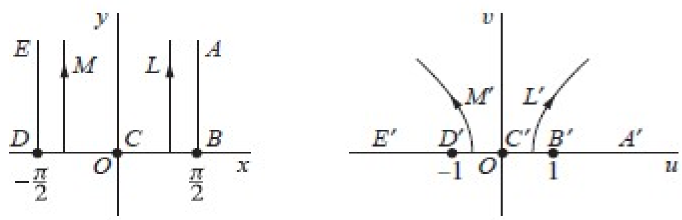

Figure-02

At the end, as a point $(-\frac{\pi}{2}, y)$ on the line segment DE moves upward from D ,its image (- coshy,0) moves from left from D′ as in directed fig-02.

Now each point in the interior -$\frac{\pi}{2} < x < \frac{\pi}{2}$,y> 0 of the strip lies on one of the vertical half lines x=c_1,y> 0 ,(-$\frac{\pi}{2} < c_1 < \frac{\pi}{2}$) that are also shown . Also it is important to note that the images of those half lines are distinct and constitute the entire half plane v≥ 0 .

This completes our presentation that the transformation w= sinz is a one one mapping of the strip -$\frac{\pi}{2} < x < \frac{\pi}{2}, y \geq 0$ onto the half plane v≥ 0.

Example- 2 let us consider anotherstrip 0<v< π as the limiting form of the rhombus with vertices at the points $w_1 = \pi i, w_{2,} w_3 = 0$ *and* w_4 *as the points* w_2 and w_4 are moved infinitely far to the left to right respectively(Fig -03).In the limit of the exterior angles become

$k_1\pi = o; k_2\pi = \pi;\ \ k_3\pi = 0$;$k_4\pi = \pi$ .We leave x_1 to be determined and chose the values x_2=0 ,x_3=1 ,x_4 =∞ .

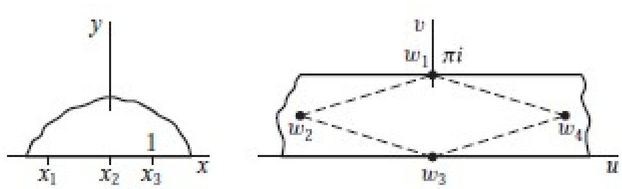

Figure- 03

Making the derivation of Schwarz-Christoffel mapping function becomes

$$\frac{dw}{dz} = A(z - x_1)^0 z^{-1}(z-1)^0$$

$$=\frac{A}{Z}$$

Then we can get W= Alogz +B

Now if B= 0 because w = 0 when z=1 .The constant A must be real because the point w lies on the real axis when z= x and x>0 .The point w = πi is the image of the point z= x_1 , where x_1 is a negative number ;

Consequently, πi = A log x_1

= A lnx_1 +A πi

By identifying real and imaginary parts here, we see that x_1 =1 and A =1

Hence the transformation becomes w = log z.

So we can understand that this transformation maps the half plane onto the strip.

The procedure used in these two examples is not in critical conditions because limiting values of angles and co ordinates were not introduced in an orderly way. Limiting values were used whenever it seemed to be appropriate to do so .But if we verify the mapping obtained, it is not essential that we justify the steps in our derivation of the mapping function. The methodwhich is used here are less tedious and simpler.

Results and Discussions

Now let us consider the two dimensional steady fluid flow between two parallel planes v=0 and v=π when the fluid is entering through a narrow slit along the line in the first plane that is perpendicular to the uv plane at the origin. Let the rate of flow of fluid into the channel through the slit be Q units of volume per unit time for each unit of depth of the channel where the depth is measured perpendicular to the uvplane. The rate of flow out at either end is then Q/2.

The transformation w=log z is a one one mapping of the upper half y >0 of the z plane onto the strip 0<v<πin the w plane (Example-2).

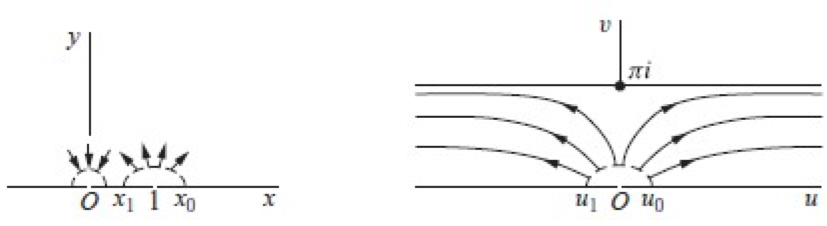

Figure - 04

The inverse transformation z = $e^w = e^u e^{iv}$ **(6)**

Maps the strip onto the half plane .Under the transformation (6) the image of the u axis is the positive of the x axis and the image of the line v=π is the negative half of the x axis. Hence the boundary of the strip is transformed into the boundary of the half plane.

Electrostatics potential about an edge of a conducting plate

Two parallel conducting plates of infinite extent are kept at the electrostatic potential V=0 and a parallel semi infinite plate placed midway between them ,is kept at the potential V=1 .

The coordinate system and the unit of length are chosen so that the plates lie in the planes v=0, v=π $and\ v = \frac{\pi}{2}$ (Fig-05). Let us find out the potential function V (u,v) in the region between those plates .

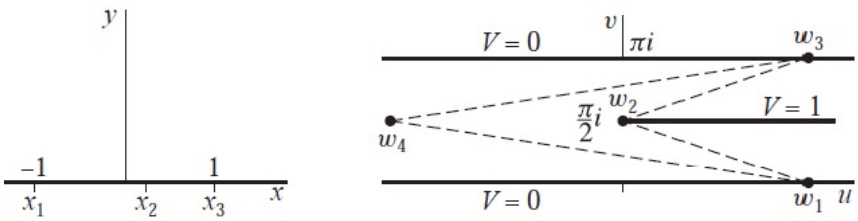

Figure -05

The cross section of that region in the uv plane has the limiting form of the quadrilateral bounded by the dashed lines in figure-05, as the point's w_1 and w_3 move out to the right and w_4 to the left. In applying the SchwarzChristoffel transformation here, we let the point x_4,corresponding to the vertex w_4 ,be the point at infinity .We choose the points x_1=-1,x_3=1 and leave x_2 to be determined . The limiting values of the exterior angles of the quadrilateral are

$$k_1\pi = \pi; k_2\pi = -\pi; \quad k_3\pi = k_4\pi = \pi$$

Thus$\frac{dw}{dz} = A(z - x_2) \quad (z+1)^{-1}(z-1)^{-1}$

$$=A\left(\frac{z-x_2}{z^2-1}\right)$$

$$=\frac{A}{2}\left(\frac{1+x_2}{z+1} + \frac{1-x_2}{z-1}\right)$$

And so the Schwarz-Christoffel transformation of the upper half of the z-plane into the divided strip in the w plane goes tothe form

$$w = \frac{A}{2}\left[(1+x_2)\log(z+1) + (1-x_2)\log(z-1)\right] + B$$

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