

Matrix Representations of Intuitionistic Fuzzy Graphs

M.G.Karunambigai*, O.K.Kalaivani*

* Department of Mathematics, Sri Vasavi College, Erode-638016,
Tamilnadu, India.

(karunsvc@yahoo.in, kalaivani83@gmail.com.)

May 23, 2016

Abstract

Matrices play an important role in the broad area of science and engineering. However, the classical matrix theory sometimes fails to solve the problems involving uncertainties, occurring in an imprecise environment. Sometimes it seems to be more natural to describe imprecise and uncertain opinions not only by membership functions and also by non membership function. In this paper, it is proved that $(CONN_{\mu(G)}(v_i, v_j), CONN_{\nu(G)}(v_i, v_j)) = (i, j)^{th}$ entry of $A + A^2 + \dots + A^{n-1}$, $\forall v_i \neq v_j \in V$, where A is the index matrix of the intuitionistic fuzzy graph G and A^k is the k^{th} power of an intuitionistic fuzzy matrix A and $(CONN_{\mu(G)}(v_i, v_j), CONN_{\nu(G)}(v_i, v_j))$ is the strength of connectedness of v_i and v_j . Also, the properties of subdivision IFG, line IFG and power of an IFG are discussed.

2010 Mathematics Subject Classification: 05C72, 03E72, 03F55.

Index Terms: incidence intuitionistic fuzzy matrix, line intuitionistic fuzzy graph, Power of an intuitionistic fuzzy graph, subdivision intuitionistic fuzzy graph.

I Introduction

Graphs can be sometimes very complicated. So one needs to find more practical ways to represent them. Matrices are a very useful way of studying graphs, since they turn the picture into numbers. Networks can represent all sorts of systems in the real world. As computers are more adept at manipulating numbers than at recognizing pictures, it is standard practice to communicate the specification of a graph to a computer in matrix form. Matrices play an important role in the broad area of science and engineering. However, the classical matrix theory sometimes fails to solve the problems involving uncertainties, occurring in an imprecise environment. Sometimes it seems to be more natural to describe imprecise and uncertain opinions not only by membership functions and also by non membership function. So an Intuitionistic fuzzy matrix is the appropriate choice when exhibiting the membership degree and non-membership degree. In 1975, Rosenfeld [17] discussed the concept of fuzzy graphs whose basic idea was introduced by Kauffmann [12] in 1973. The fuzzy relations between fuzzy sets were also considered by Rosenfeld and he developed the structure of fuzzy graphs, obtaining analogs of several

graph theoretical concepts. The first definition of fuzzy graph was introduced by Kaufman[12] in 1973, based on Zadeh's fuzzy relations in 1971[20]. Atanassov[3][19] introduced the concept of intuitionistic fuzzy(IF) relations and intuitionistic fuzzy graphs(IFGs). M.G.Karunambigai and R.Parvathi[10][14] introduced the concept of IFG elaborately and analysed its components. Atanassov introduced the index matrix representation of intuitionistic fuzzy graphs and discussed its operations in [5][4][6]. Akram et al. discussed the properties of strong intuitionistic fuzzy graphs and also the properties of intuitionistic fuzzy cycle and intuitionistic fuzzy trees in [1][2]. R.Parvathi et al.[16] discussed operations on intuitionistic fuzzy graphs using index matrices. Intuitionistic fuzzy matrices are extensively used for decision making problems, cluster analysis, pattern recognition, medical diagnosis and network problems. Intuitionistic fuzzy matrices can be used whenever uncertainty occurs in a problem. These application motivated us to consider intuitionistic fuzzy matrices and discuss its properties. The paper is organized as follows. In section 2, we review the basic definitions of intuitionistic fuzzy graph. Section 3 deals with the properties of the power of an intuitionistic fuzzy graph and given the relationship between the index matrix of an intuitionistic fuzzy graph and power of an intuitionistic fuzzy graph and section 4 concludes the paper.

II Preliminaries

In this section, the basic definitions and Theorems which are used to prove the forthcoming results are given.

Definition 2.1 [8] *A crisp graph $G^* = (V, E)$ is an ordered triple $(V(G^*), E(G^*), \psi_{G^*})$ consisting of a non-empty $V(G^*)$ of vertices, a set $E(G^*)$, disjoint from $V(G^*)$, of edges and an incidence function ψ_{G^*} that associates with each edge of G^* an unordered pair of vertices of G^* .*

Definition 2.2 [8] *Let $G^* = (V, E)$ be a crisp graph. A walk is a sequence of vertices and edges, where the endpoints of each edge are the preceding and following vertices in the sequence. A path is a walk without repeated vertices. If a walk (resp. trail, path) begins at u and ends at v then it is an $u - v$ walk. A walk is closed if it begins and ends at the same vertex.*

Definition 2.3 [8] *Let $G^* = (V, E)$ be a crisp graph. The length of a path $P = v_1v_2\dots v_{n+1}$ in G is n .*

Definition 2.4 [8] *Let $G^* = (V, E)$ be a crisp graph. The distance between the two vertices v_i and v_j in G^* is denoted by $d_{G^*}(v_i, v_j)$ and is defined as the minimum length of the path connecting the vertices v_i and v_j .*

Definition 2.5 *A matrix is a rectangular array of numbers arranged in rows and columns. The number of rows and columns that a matrix has, called its dimension or its order. That is, the dimension or order of a matrix with m rows and n columns is $m \times n$. The individual items in a matrix are called its elements or entries.*

Definition 2.6 *Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices. Then two matrices A and B are equal to each other, if they have the same dimensions $m \times n$ and the same elements $a_{ij} = b_{ij}$ for $i = 1, \dots, n$ and $j = 1, \dots, m$. It is denoted by $A = B$*

Definition 2.7 [10] *An Intuitionistic Fuzzy Graph (IFG) is of the form $G = (V, E)$ said to be a minmax IFG if*

- (1) $V = \{v_1, \dots, v_n\}$ such that $\mu_i : V \rightarrow [0, 1]$ and $\nu_i : V \rightarrow [0, 1]$, denotes the degree of membership and non-membership of an element $v_i \in V$ respectively and $0 \leq \mu_i + \nu_i \leq 1$, for every $v_i \in V$,
- (2) $E \subseteq V \times V$ where $\mu_{ij} : V \times V \rightarrow [0, 1]$ and $\nu_{ij} : V \times V \rightarrow [0, 1]$ are such that
- $$\mu_{ij} \leq \min(\mu_i, \mu_j)$$
- $$\nu_{ij} \leq \max(\nu_i, \nu_j),$$
- denotes the degree of membership and non-membership of an edge $e_{ij} = (v_i, v_j) \in E$ respectively, where, $0 \leq \mu_{ij} + \nu_{ij} \leq 1$, for every $e_{ij} = (v_i, v_j) \in E$. The degree of hesitance(hesitation degree) of the vertex $v_i \in V$ in G is $\Pi_i = 1 - \mu_i - \nu_i$ and the degree of hesitance(hesitation degree) of an edge $e_{ij} = (v_i, v_j) \in E$ in G is $\Pi_{ij} = 1 - \mu_{ij} - \nu_{ij}$.

Definition 2.8 [10] Let $G = (V, E)$ be an intuitionistic fuzzy graph. A walk is a sequence of vertices and edges, where the endpoints of each edge are the preceding and following vertices in the sequence, such that either one of the following conditions is satisfied.

- 1) $\mu_{ij} > 0$ & $\nu_{ij} = 0$ for some i & j . 2) $\mu_{ij} > 0$ & $\nu_{ij} > 0$ for some i & j . If a walk begins at v_i and ends at v_j then it is an $v_i - v_j$ walk. A walk is closed if it begins and ends at the same vertex.

Definition 2.9 [10] Let $G = (V, E)$ be an intuitionistic fuzzy graph. A path P in an intuitionistic fuzzy graph G is a sequence of distinct vertices v_1, v_2, \dots, v_n such that either one of the following conditions is satisfied.

- 1) $\mu_{ij} > 0$ & $\nu_{ij} = 0$ for some i & j . 2) $\mu_{ij} > 0$ & $\nu_{ij} > 0$ for some i & j .

Definition 2.10 [10] Let $G = (V, E)$ be an intuitionistic fuzzy graph. The length of a path $P = v_1 v_2 \dots v_{n+1}$ ($n > 0$) in G is n .

Definition 2.11 [10] An intuitionistic fuzzy graph $G = (V, E)$ is connected if any two vertices are joined by a path.

Definition 2.12 [10] Let $G = (V, E)$ be an intuitionistic fuzzy graph. The μ - strength of a path $P = v_1 v_2 \dots v_n$ in an intuitionistic fuzzy graph G is denoted by $S_{\mu(G)}(P)$ and is defined as $\min\{\mu_{ij}\}$, for all $i, j = 1, 2, \dots, n$

Definition 2.13 [10] Let $G = (V, E)$ be an intuitionistic fuzzy graph. The ν - strength of a path $P = v_1 v_2 \dots v_n$ in an intuitionistic fuzzy graph G is denoted by $S_{\nu(G)}(P)$ and is defined as $\max\{\nu_{ij}\}$, for all $i, j = 1, 2, \dots, n$

Definition 2.14 [10] If $v_i, v_j \in V \subseteq G$, the μ - strength of connectedness between the vertices v_i and v_j in G is $CONN_{\mu(G)}(v_i, v_j) = \max\{S_{\mu(G)}(P)\}$ and ν - strength of connectedness between the vertices v_i and v_j in G is $CONN_{\nu(G)}(v_i, v_j) = \min\{S_{\nu(G)}(P)\}$ for all possible paths between v_i and v_j .

Definition 2.15 [10] An intuitionistic fuzzy graph, $G = (V, E)$ is said to be a strong intuitionistic fuzzy graph if

$$\mu_{ij} = \min(\mu_i, \mu_j) \text{ and } \nu_{ij} = \max(\nu_i, \nu_j), \forall (v_i, v_j) \in E.$$

Definition 2.16 [10] An intuitionistic fuzzy graph, $G = (V, E)$ is said to be a complete intuitionistic fuzzy graph if

$$\mu_{ij} = \min(\mu_i, \mu_j) \text{ and } \nu_{ij} = \max(\nu_i, \nu_j), \forall v_i, v_j \in V.$$

Definition 2.17 [13] *The order of an intuitionistic fuzzy graph $G = (V, E)$ is defined as $O(G) = (O_\mu(G), O_\nu(G))$ where*

$$O_\mu(G) = \sum_{v_i \in V} \mu_i \text{ and } O_\nu(G) = \sum_{v_i \in V} \nu_i$$

Definition 2.18 [13] *The size of an intuitionistic fuzzy graph is defined as $S(G) = (S_\mu(G), S_\nu(G))$*

$$S_\mu(G) = \sum_{e_{ij} \in E} \mu_{ij} \text{ and } S_\nu(G) = \sum_{e_{ij} \in E} \nu_{ij}$$

Definition 2.19 [13] *Let $G = (V, E)$ be an intuitionistic fuzzy graph. The neighbourhood of a vertex $v_i \in V$ is denoted by $N_G(v_i)$ and is defined as $N_G(v_i) = \{v_j \in V | (v_i, v_j) \in E\}$.*

Definition 2.20 *The function $f : X \rightarrow Y$ is an one to one function if and only if for every element $y \in Y$ there is exactly one element $x \in X$. In Symbol, $f(x) = f(y) \Rightarrow x = y, \forall x, y \in X$.*

Definition 2.21 *The function $f : X \rightarrow Y$ is an onto function if and only if for every element $y \in Y$ there is at least one element $x \in X$. In Symbol, $f(x) = y, \forall y \in Y$.*

Definition 2.22 *A function $f : X \rightarrow Y$ is a bijection if the function is both one-one and onto mapping of a set X to a set Y .*

Definition 2.23 [11] *A homomorphism from a intuitionistic fuzzy graph $G_1 = (V_1, E_1)$ to a intuitionistic fuzzy graph $G_2 = (V_2, E_2)$, written $f : G_1 \rightarrow G_2$, is a mapping $f : V_1 \rightarrow V_2$ from the vertex set of G_1 to the vertex set of G_2 such that if any two vertices $v_i, v_j \in V_1$ are adjacent in G_1 , then $f(v_i), f(v_j) \in V_2$ are adjacent in G_2 and*

$$\begin{aligned} \mu(v_i) &\leq \mu'(f(v_i)) \text{ and } \nu(v_i) \geq \nu'(f(v_i)), \forall v_i \in V_1 \\ \mu(v_i, v_j) &\leq \mu'(f(v_i), f(v_j)) \text{ and } \nu(v_i, v_j) \geq \nu'(f(v_i), f(v_j)), \forall v_i, v_j \in V_1. \end{aligned}$$

Definition 2.24 [11] *Two intuitionistic fuzzy graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic if there is a bijections $f : V_1 \rightarrow V_2$ such that any two vertices $v_i, v_j \in V_1$ are adjacent in G_1 if and only if $f(v_i), f(v_j) \in V_2$ are adjacent in G_2 and*

$$\begin{aligned} \mu(v_i) &= \mu'(f(v_i)) \text{ and } \nu(v_i) = \nu'(f(v_i)), \forall v_i \in V_1 \\ \mu(v_i, v_j) &= \mu'(f(v_i), f(v_j)) \text{ and } \nu(v_i, v_j) = \nu'(f(v_i), f(v_j)), \forall v_i, v_j \in V_1. \end{aligned}$$

Definition 2.25 [11] *Two intuitionistic fuzzy graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be co-weak isomorphic if there is a bijections $f : V_1 \rightarrow V_2$ such that any two vertices $v_i, v_j \in V_1$ are adjacent in G_1 if and only if $f(v_i), f(v_j) \in V_2$ are adjacent in G_2 and*

$$\begin{aligned} \mu(v_i) &\leq \mu'(f(v_i)) \text{ and } \nu(v_i) \geq \nu'(f(v_i)), \forall v_i \in V_1 \\ \mu(v_i, v_j) &= \mu'(f(v_i), f(v_j)) \text{ and } \nu(v_i, v_j) = \nu'(f(v_i), f(v_j)), \forall v_i, v_j \in V_1 \end{aligned}$$

Definition 2.26 [9] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two IFGs with $V_1 \cap V_2 \neq \phi$. Then the union of G_1 and G_2 is an IFG, denoted by $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ and is defined as

$$((\mu \cup \mu')(v_i), (\nu \cup \nu')(v_i)) = \begin{cases} (\mu_i, \nu_i), & \text{if } v_i \in V_1 - V_2, \\ (\mu'_i, \nu'_i), & \text{if } v_i \in V_2 - V_1, \\ (\max(\mu_i, \mu'_i), \min(\nu_i, \nu'_i)) & \text{if } v_i \in V_1 \cap V_2. \end{cases}$$

$$(\mu \cup \mu')(v_i, v_j) = \begin{cases} \mu_{ij} & \text{if } e_{ij} \in E_1 - E_2, \\ \mu_{ij}' & \text{if } e_{ij} \in E_2 - E_1, \\ \max(\mu_{ij}, \mu'_{ij}) & \text{if } e_{ij} \in E_1 \cap E_2, \\ \min((\mu_i \cup \mu'_i), \max(\mu_j, \mu'_j)) & \text{if } v_i \in V_1 - V_2, \\ & v_j \in V_1 \cap V_2 \text{ and} \\ & e_{ij} \in E_1 - E_2 \\ & \text{or } e_{ij} \in E_2 - E_1, \\ (0, 1) & \text{otherwise.} \end{cases}$$

$$(\nu \cup \nu')(v_i, v_j) = \begin{cases} \nu_{ij} & \text{if } e_{ij} \in E_1 - E_2, \\ \nu_{ij}' & \text{if } e_{ij} \in E_2 - E_1, \\ \min((\nu_i \cup \nu'_i), (\nu_j \cup \nu'_j)) & \text{if } e_{ij} \in E_1 \cap E_2, \\ \max((\nu_i \cup \nu'_i), \min(\nu_j, \nu'_j)) & \text{if } v_i \in V_1 - V_2, \\ & v_j \in V_1 \cap V_2 \text{ and} \\ & e_{ij} \in E_1 - E_2 \\ & \text{or } e_{ij} \in E_2 - E_1, \\ (0, 1) & \text{otherwise.} \end{cases}$$

Definition 2.27 [15] An intuitionistic fuzzy matrix(IFM) is a matrix of order $m \times n$ and is defined as $A = \{ \langle a_{\mu_{ij}}, a_{\nu_{ij}} \rangle \}_{m \times n}$, where $a_{\mu_{ij}} \in [0, 1]$, $a_{\nu_{ij}} \in [0, 1]$ such that $0 \leq a_{\mu_{ij}} + a_{\nu_{ij}} \leq 1$, $1 \leq i \leq m$

and $1 \leq j \leq n$. It can also be represented in the matrix form,

$$A = \{ \langle a_{\mu_{ij}}, a_{\nu_{ij}} \rangle \}_{m \times n} = \begin{pmatrix} \langle a_{\mu_{11}}, a_{\nu_{11}} \rangle & \langle a_{\mu_{12}}, a_{\nu_{12}} \rangle & \dots & \langle a_{\mu_{1n}}, a_{\nu_{1n}} \rangle \\ \langle a_{\mu_{21}}, a_{\nu_{21}} \rangle & \langle a_{\mu_{22}}, a_{\nu_{22}} \rangle & \dots & \langle a_{\mu_{2n}}, a_{\nu_{2n}} \rangle \\ \dots & \dots & \dots & \dots \\ \langle a_{\mu_{m1}}, a_{\nu_{m1}} \rangle & \langle a_{\mu_{m2}}, a_{\nu_{m2}} \rangle & \dots & \langle a_{\mu_{mn}}, a_{\nu_{mn}} \rangle \end{pmatrix}$$

Definition 2.28 The number of rows and columns that IF matrix has called its dimension or its order. That is, the dimension or order of IF matrix with m rows and n columns is $m \times n$. The individual items in an IF matrix are called its elements or entries.

Definition 2.29 [15] Let $A = \{ \langle a_{\mu_{ij}}, a_{\nu_{ij}} \rangle \}_{m \times n}$ be a intuitionistic fuzzy matrix. The transpose of the matrix A is denoted by A^T and is defined as $A^T = \{ \langle a_{\mu_{ji}}, a_{\nu_{ji}} \rangle \}_{n \times m}$.

Definition 2.30 Let $A = \{ \langle a_{\mu_{ij}}, a_{\nu_{ij}} \rangle \}_{m \times n}$ and $B = \{ \langle b_{\mu_{ij}}, b_{\nu_{ij}} \rangle \}_{m \times n}$ be two intuitionistic fuzzy matrices. Then two IF matrices A and B are equal to each other, if they have the same dimensions $m \times n$ and the same elements $a_{\mu_{ij}} = b_{\mu_{ij}}, a_{\nu_{ij}} = b_{\nu_{ij}}$ for $i = 1, \dots, m$ and $j = 1, \dots, n$. It is denoted by $A = B$

Definition 2.31 Let $A = \{ \langle a_{\mu_{ij}}, a_{\nu_{ij}} \rangle \}_{m \times n}$ and $B = \{ \langle b_{\mu_{ij}}, b_{\nu_{ij}} \rangle \}_{m \times n}$ be two intuitionistic fuzzy matrices, then the sum of A and B is denoted by $A +_{\max - \min} B$, is defined as $A +_{\max - \min} B = \{ \langle c_{\mu_{ij}}, c_{\nu_{ij}} \rangle \}_{m \times n} = [\langle \max(a_{\mu_{ij}}, b_{\mu_{ij}}), \min(a_{\nu_{ij}}, b_{\nu_{ij}}) \rangle]$, $1 \leq i \leq m, 1 \leq j \leq n$.

Notation 2.1 Throughout this paper, we denote $''+''_{\max - \min}$ as $''+''$.

Theorem 2.2 [8] If a crisp graph G^* contains a $u - v$ walk of length l , then G^* contains a $u - v$ path of length l .

Theorem 2.3 [8] Let G^* be a crisp graph. Then G^* is connected if and only if every pair of vertices in G^* is connected.

III Matrix Representations of Intuitionistic Fuzzy Graphs

In this section, the properties of the power of an intuitionistic fuzzy graph and the relationship between the index matrix of an intuitionistic fuzzy graph and power of an intuitionistic fuzzy graph have been analysed.

Definition 3.32 Let $A = \{ \langle a_{\mu_{ij}}, a_{\nu_{ij}} \rangle \}_{m \times n}$ and $B = \{ \langle b_{\mu_{ij}}, b_{\nu_{ij}} \rangle \}_{n \times p}$ be two intuitionistic fuzzy matrices. Then the two types of product of A and B are defined as

1) $\max - \min$ product of IF matrices : $A \bullet_{\max - \min} B = \{ \langle c_{\mu_{ij}}, c_{\nu_{ij}} \rangle \}_{m \times p} = [\langle \max(\min(a_{\mu_{ij}}, b_{\mu_{jk}})), \min(\max(a_{\nu_{ij}}, b_{\nu_{jk}})) \rangle]$, $1 \leq i \leq m, 1 \leq j \leq p, 1 \leq k \leq n$ and

2) $\min - \max$ product of IF matrices: $A \bullet_{\min - \max} B = \{ \langle c_{\mu_{ij}}, c_{\nu_{ij}} \rangle \}_{m \times p} = [\langle \min(\max(a_{\mu_{ij}}, b_{\mu_{jk}})), \max(\min(a_{\nu_{ij}}, b_{\nu_{jk}})) \rangle]$, $1 \leq i \leq m, 1 \leq j \leq p, 1 \leq k \leq n$.

Definition 3.33 Let $A = \{ \langle a_{\mu_{ij}}, a_{\nu_{ij}} \rangle \}_{m \times n}$ be intuitionistic fuzzy matrix and k is a positive integer. Then the k^{th} power of an intuitionistic fuzzy matrix is denoted by A^k and is defined as max – min product of k –copies of an intuitionistic fuzzy matrix A .

Definition 3.34 [5] Let $G = (V, E)$ be an intuitionistic fuzzy graph. The index matrix representation of intuitionistic fuzzy graph(IMIFG) is of the form $[V, E \subset V \times V]$ where $V = \{v_1, v_2, \dots, v_n\}$ and

$$E = \{ \langle \mu_{ij}, \nu_{ij} \rangle \}_{m \times n} = \begin{matrix} & v_1 & v_2 & \dots & v_n \\ \begin{matrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{matrix} & \left(\begin{array}{cccc} \langle \mu_{11}, \nu_{11} \rangle & \langle \mu_{12}, \nu_{12} \rangle & \dots & \langle \mu_{1n}, \nu_{1n} \rangle \\ \langle \mu_{21}, \nu_{21} \rangle & \langle \mu_{22}, \nu_{22} \rangle & \dots & \langle \mu_{2n}, \nu_{2n} \rangle \\ \dots & & & \\ \langle \mu_{n1}, \nu_{n1} \rangle & \langle \mu_{n2}, \nu_{n2} \rangle & \dots & \langle \mu_{nn}, \nu_{nn} \rangle \end{array} \right) \end{matrix}$$

where $\langle \mu_{ij}, \nu_{ij} \rangle \in [0, 1] \times [0, 1] (1 \leq i, j \leq n)$, the edge between two vertices v_i and v_j is indexed by $\langle \mu_{ij}, \nu_{ij} \rangle$.

Note 3.4 Index matrix representation of any intuitionistic fuzzy graph is an intuitionistic fuzzy matrix.

Definition 3.35 Let $G = (V, E)$ be an intuitionistic fuzzy graph where $V = \{v_1, v_2, \dots, v_n\}$. The incidence matrix of an intuitionistic fuzzy graph G is $B = \{ \langle b_{\mu_{ij}}, b_{\nu_{ij}} \rangle \}_{n \times m}$, where n and m represents the number of vertices and number of edges of G respectively, whose entries of B are as follows:

$$B = \{ \langle b_{\mu_{ij}}, b_{\nu_{ij}} \rangle \}_{n \times m} = \begin{cases} \langle \mu(e_j), \nu(e_j) \rangle, & \text{if an edge } e_j \text{ is incident on the vertex } v_i \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases}$$

It can also be represented in the matrix form,

$$B = \{ \langle b_{\mu_{ij}}, b_{\nu_{ij}} \rangle \}_{n \times m} = \begin{matrix} & e_1 & e_2 & \dots & e_n \\ \begin{matrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{matrix} & \left(\begin{array}{cccc} \langle \mu(e_1), \nu(e_1) \rangle & \langle \mu(e_2), \nu(e_2) \rangle & \dots & \langle \mu(e_n), \nu(e_n) \rangle \\ \langle \mu(e_1), \nu(e_1) \rangle & \langle \mu(e_2), \nu(e_2) \rangle & \dots & \langle \mu(e_n), \nu(e_n) \rangle \\ \dots & & & \\ \langle \mu(e_1), \nu(e_1) \rangle & \langle \mu(e_2), \nu(e_2) \rangle & \dots & \langle \mu(e_n), \nu(e_n) \rangle \end{array} \right) \end{matrix}$$

where $\langle \mu(e_j), \nu(e_j) \rangle \in [0, 1] \times [0, 1]$.

Definition 3.36 Let $G = (V, E)$ be an intuitionistic fuzzy graph, where $V = \{v_1, v_2, \dots, v_n\}$ and $E = \{e_1, e_2, \dots, e_k\}$. Then the line intuitionistic fuzzy graph is denoted by $G_L = (V_L, E_L)$, where the vertices of G_L are in one-one correspondence with the edges of G and there exist an edge between the vertices of G_L if and only if the corresponding edges of G are adjacent. The membership and non-membership value of V_L and E_L are defined as follows:

$$\mu_L(v_i) = \mu(e_i) \text{ and } \nu_L(v_i) = \nu(e_i), \forall e_i \in E.$$

$$\mu_L(v_i, v_j) = \begin{cases} \min(\mu_L(v_i), \mu_L(v_j)), & \text{if } e_i \text{ and } e_j \text{ are adjacent in } G \\ (0, 1), & \text{otherwise} \end{cases}$$

$$\nu_L(v_i, v_j) = \begin{cases} \max(\nu_L(v_i), \nu_L(v_j)), & \text{if } e_i \text{ and } e_j \text{ are adjacent in } G \\ (0, 1), & \text{otherwise} \end{cases}$$

Example 3.1 Consider an intuitionistic fuzzy graph, $G = (V, E)$, such that $V = \{v_1, v_2, v_3, v_4\}$, $E = \{(v_1, v_2), (v_1, v_3), (v_2, v_3), (v_3, v_4), (v_4, v_1)\}$ and $V_L = \{v_{12}, v_{23}, v_{34}, v_{14}, v_{13}\}$ and $E_L = \{(v_{12}, v_{23}), (v_{23}, v_{34}), (v_{34}, v_{14}), (v_{14}, v_{12}), (v_{12}, v_{13}), (v_{14}, v_{13}), (v_{13}, v_{23}), (v_{13}, v_{34})\}$

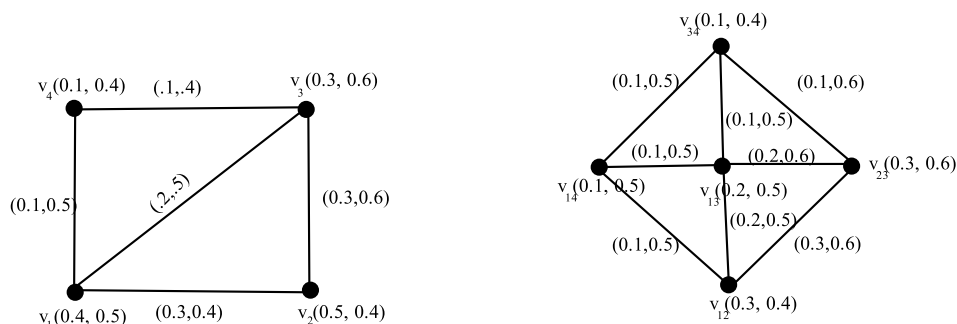


Figure 1: G and G_L

Definition 3.37 Let $G = (V, E)$ be an intuitionistic fuzzy graph with the underlying crisp graph $G^* = (V, E)$. Then the subdivision of an intuitionistic fuzzy graph G is denoted by $G_{sd} = (V_{sd}, E_{sd})$ and is obtained by adding a new vertex u_k into every edge $e_{ij} = (v_i, v_j) \in E$ of G such that the membership and the non-membership of the vertex v_k and the edges $v_i u_k$ and $u_k v_j$ are defined as follows:

$$\begin{aligned} \mu_{sd}(u_k) &= \mu_{ij} \text{ and } \nu_{sd}(u_k) = \nu_{ij}, \forall u_k \in V_{sd} \\ \mu_{sd}(v_i, u_k) &\leq \min(\mu_{sd}(v_i), \mu_{sd}(u_k)) \text{ and } \nu_{sd}(v_i, u_k) \leq \max(\nu_{sd}(v_i), \nu_{sd}(u_k)) \\ \mu_{sd}(u_k, v_j) &\leq \min(\mu_{sd}(u_k), \mu_{sd}(v_j)) \text{ and } \nu_{sd}(u_k, v_j) \leq \max(\nu_{sd}(u_k), \nu_{sd}(v_j)), \forall v_i, v_j, u_k \in V_{sd}. \end{aligned}$$

Example 3.2 Consider an intuitionistic fuzzy graph, $G = (V, E)$, such that $V = \{v_1, v_2, v_3, v_4, v_5\}$, $E = \{(v_1, v_2), (v_1, v_5), (v_2, v_3), (v_2, v_4), (v_3, v_4), (v_4, v_5)\}$ and $V_{sd} = \{v_1, v_2, v_3, v_4, v_5, u_1, u_2, u_3, u_4, u_5\}$ and $E_{sd} = \{(v_1, u_1), (u_1, v_2), (v_2, u_2), (u_1, v_3), (v_3, u_3), (u_3, v_4), (v_4, u_4), (u_4, v_5), (v_5, u_5), (u_5, v_1)\}$

Figure 2: G and G_{sd}

Definition 3.38 Let $G = (V, E)$ be an intuitionistic fuzzy graph with the underlying crisp graph $G^* = (V, E)$ where $V = \{v_1, v_2, \dots, v_n\}$. Then the power of an intuitionistic fuzzy graph G is denoted by, $G^k = (V^k, E^k)$, where $V^k = V$ and the vertices v_i and v_j are adjacent in G^k if and only if $d_{G^*}(v_i, v_j) \leq k$ (Refer Definition 1.4). The membership and non-membership values of the edges of G^k are defined as follows:

$$(\mu^k(v_i, v_j), \nu^k(v_i, v_j)) = \begin{cases} (\min(\mu_i, \mu_j), \max(\nu_i, \nu_j)), & \text{if } d_{G^*}(v_i, v_j) \leq k \\ (0, 1), & \text{otherwise} \end{cases}$$

Example 3.3 Consider an intuitionistic fuzzy graph, $G = (V, E)$, such that $V = \{v_1, v_2, v_3, v_4, v_5\}$, $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5)\}$, $E^2 = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5), (v_1, v_3), (v_3, v_5), (v_2, v_4), (v_1, v_4), (v_2, v_5)\}$ and $E^4 = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5), (v_1, v_3), (v_3, v_5), (v_2, v_4), (v_1, v_4), (v_2, v_5), (v_1, v_5)\}$

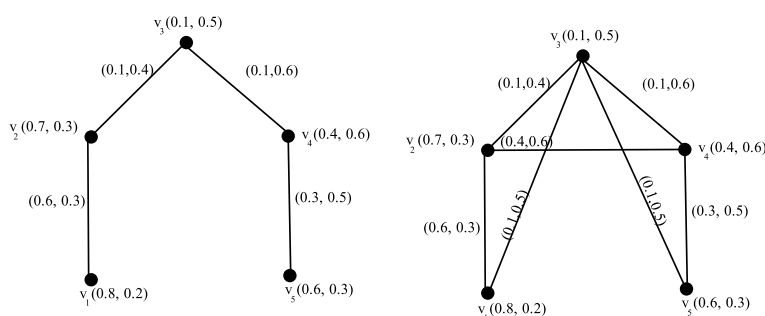


Figure 3: G and G^2

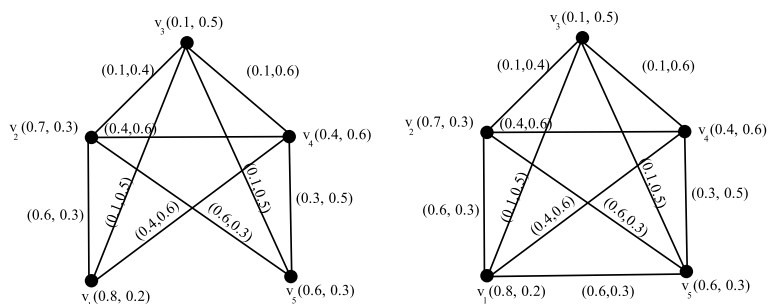


Figure 4: G^3 and G^4

It can also be represented in the matrix form,

$$(i, j)^{th} \text{ entry of } A = \begin{matrix} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} & \begin{bmatrix} \langle 0, 1 \rangle & \langle .6, .3 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle \\ \langle .6, .3 \rangle & \langle 0, 1 \rangle & \langle .1, .4 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle \\ \langle 0, 1 \rangle & \langle .1, .4 \rangle & \langle 0, 1 \rangle & \langle .1, .6 \rangle & \langle 0, 1 \rangle \\ \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle .1, .6 \rangle & \langle 0, 1 \rangle & \langle .3, .5 \rangle \\ \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle .3, .5 \rangle & \langle 0, 1 \rangle \end{bmatrix} \end{matrix}$$

$$(i, j)^{th} \text{ entry of } A^2 = \begin{matrix} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} & \begin{bmatrix} \langle .6, .3 \rangle & \langle 0, 1 \rangle & \langle .1, .4 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle \\ \langle 0, 1 \rangle & \langle .6, .3 \rangle & \langle 0, 1 \rangle & \langle .1, .6 \rangle & \langle 0, 1 \rangle \\ \langle .1, .4 \rangle & \langle 0, 1 \rangle & \langle .1, .4 \rangle & \langle 0, 1 \rangle & \langle .1, .6 \rangle \\ \langle 0, 1 \rangle & \langle .1, .6 \rangle & \langle 0, 1 \rangle & \langle .3, .5 \rangle & \langle 0, 1 \rangle \\ \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle .1, .6 \rangle & \langle 0, 1 \rangle & \langle .3, .5 \rangle \end{bmatrix} \end{matrix}$$

Theorem 3.5 Let $G = (V, E)$ be a intuitionistic fuzzy graph. If an intuitionistic fuzzy graph G contains a $u - v$ walk of length l , then G contains a $u - v$ path of length l .

Proof. Proof follows from the Definition 1.8 and Theorem 2.2. ■

Theorem 3.6 Let $G = (V, E)$ be a strong intuitionistic fuzzy graph and $G_{sd} = (V_{sd}, E_{sd})$ be the subdivision

of an intuitionistic fuzzy graph G . Then $S_\mu(G_{sd}) \leq 2S_\mu(G)$ and $S_\nu(G_{sd}) \leq 2S_\nu(G)$, where S_μ and S_ν are size of G .

Proof. Let $G = (V, E)$ be a strong intuitionistic fuzzy graph and $G_{sd} = (V_{sd}, E_{sd})$ be the subdivision of an intuitionistic fuzzy graph G . The size of G is $S(G) = (S_\mu(G), S_\nu(G))$, where Consider,

$$\begin{aligned} S_\mu(G_{sd}) &= \sum_{e_{ik}, e_{kj} \in E_{sd}} (\mu_{sd}(e_{ik}) + \mu_{sd}(e_{kj})), \\ &\leq \sum \mu_{sd}(v_k) + \sum \mu_{sd}(v_k), \text{ since } G \text{ is strong} \\ &\leq \sum \mu_{ij} + \sum \mu_{ij}, \text{ since by Definition 1.37} \\ &\leq 2 \sum \mu_{ij} \\ &\leq 2S_\mu(G). \end{aligned}$$

and

$$\begin{aligned} S_\nu(G_{sd}) &= \sum_{e_{ik}, e_{kj} \in E_{sd}} (\nu_{sd}(e_{ik}) + \nu_{sd}(e_{kj})) \\ &\leq \sum \nu_{sd}(v_k) + \sum \nu_{sd}(v_k), \text{ since } G \text{ is strong} \\ &\leq \sum \nu_{ij} + \sum \nu_{ij}, \text{ since by Definition 1.37} \\ &\leq 2 \sum \nu_{ij} \\ &\leq 2S_\nu(G). \end{aligned}$$

■

Theorem 3.7 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two strong IFGs. Then $A(G_1 \cup G_2) = A(G_1) + A(G_2)$ if and only if $V_1 = V_2$.

Proof. Let us assume that $A(G_1 \cup G_2) = A(G_1) + A(G_2)$. Suppose that G_1 and G_2 are having different vertex set and let $V_1 = \{v_1, v_2, \dots, v_m\}$ and $V_2 = \{u_1, u_2, \dots, u_n\}$, where $v_i \neq u_j, i = 1, 2, \dots, m, j = 1, 2, \dots, n, \forall i$ and j

Case(i): Let $m \neq n$. Then $V(G_1 \cup G_2) = \{v_1, v_2, \dots, v_m, u_1, u_2, \dots, u_n\}$ and the order of the IF matrix $A(G_1 \cup G_2) = m + n \times m + n$. But the order of the IF matrix $A(G_1) = m \times m$ and the order of the IF matrix $A(G_2) = n \times n$. Therefore $A(G_1) + A(G_2)$ is not possible, since by Definition 1.31, the order of the matrices $A(G_1)$ and $A(G_2)$ are not equal. Hence by Definition 1.30, $A(G_1 \cup G_2) \neq A(G_1) + A(G_2)$, which is a contradiction to our assumption that $A(G_1 \cup G_2) = A(G_1) + A(G_2)$. Hence $V_1 = V_2$.

Case(ii): Let $m = n$. Then $V(G_1 \cup G_2) = V_1 \cup V_2 = \{v_1, v_2, \dots, v_m, u_1, u_2, \dots, u_m\}$ and the order of the matrix $A(G_1 \cup G_2) = 2m \times 2m$. But the order of the IF matrix $A(G_1) = m \times m, A(G_2) = m \times m$. Therefore the order of the IF matrix $A(G_1) + A(G_2) = m \times m \neq A(G_1 \cup G_2)$, which is contradiction to our assumption that $A(G_1 \cup G_2) = A(G_1) + A(G_2)$. Hence $V_1 = V_2$.

Conversely, Let us assume that $V_1 = V_2$. and let $V_1 = \{v_1, v_2, \dots, v_m\}$ and $V_2 = \{v_1, v_2, \dots, v_m\}$. Then the order of IF matrix $A(G_1 \cup G_2) = m \times m = A(G_1) + A(G_2)$. Next, in order to prove the entries of the IF matrix $A(G_1 \cup G_2) =$ the entries of the IF matrix $A(G_1) + A(G_2)$, we need to consider the following three subcases:

Subcase(i): Let $(v_i, v_j) \in E_1 \cap E_2$

By Definition 1.26 ,

$$\begin{aligned}
 (\mu \cup \mu')(v_i, v_j) &= \max(\mu_{ij}, \mu'_{ij}) \\
 (\nu \cup \nu')(v_i, v_j) &= \min((\nu_i \cup \nu'_i), (\nu_j \cup \nu'_j)) \\
 &= \min(\min(\nu_i, \nu'_i), \min(\nu_j, \nu'_j)) \\
 &= \min(\min(\nu_i, \nu'_i), \min(\nu'_i, \nu'_j)), \text{ Since } G_1 \text{ \& } G_2 \text{ are strong IFGs} \\
 (\nu \cup \nu')(v_i, v_j) &= \min(\nu_{ij}, \nu'_{ij})
 \end{aligned}$$

Therefore, $(i, j)^{th}$ entry of

$$A(G_1 \cup G_2) = (\max(\mu_{ij}, \mu'_{ij}), \min(\nu_{ij}, \nu'_{ij})) \tag{1}$$

By Definition 1.31 and 1.34, $(i, j)^{th}$ entry of

$$A(G_1) + A(G_2) = (\max(\mu_{ij}, \mu'_{ij}), \min(\nu_{ij}, \nu'_{ij})) \tag{2}$$

From Equation (1) and (2), $(i, j)^{th}$ entry of $A(G_1 \cup G_2) = (i, j)^{th}$ entry of $A(G_1) + A(G_2)$

Subcase(ii): Let $e_{ij} \in E_1 - E_2$. Then by Definition 1.26, $(\mu \cup \mu')(v_i, v_j) = \mu_{ij}$ and $(\nu \cup \nu')(v_i, v_j) = \nu_{ij}$. Therefore, $(i, j)^{th}$ entry of

$$A(G_1 \cup G_2) = (\mu_{ij}, \nu_{ij}) \tag{3}$$

If $e_{ij} \in E_1$, then $(\mu_{ij}, \nu_{ij}) \neq (0, 1)$ and if $e_{ij} \notin E_2$, then $(\mu'_{ij}, \nu'_{ij}) = (0, 1)$.

Therefore, $(i, j)^{th}$ entry of

$$A(G_1) + A(G_2) = (\max(\mu_{ij}, \mu'_{ij}), \min(\nu_{ij}, \nu'_{ij})) = (\mu_{ij}, \nu_{ij}) \tag{4}$$

From Equation (3) and (4), $(i, j)^{th}$ entry of $A(G_1 \cup G_2) = (i, j)^{th}$ entry of $A(G_1) + A(G_2)$.

Subcase(iii): Let $e_{ij} \in E_2 - E_1$. Then proof follows from the Subcase(ii) by replacing E_1 by E_2 and E_2 by E_1 . Hence, from the Subcases (i) – (iii), it follows that $(i, j)^{th}$ entry of $A(G_1 \cup G_2) = (i, j)^{th}$ entry of $A(G_1) + A(G_2)$.

Theorem 3.8 Let $G = (V, E)$ be an IFG and let $A = \{ \langle \mu_{ij}, \nu_{ij} \rangle \}$ be the index matrix of G . Then for each positive integer k , the

$$(i, j)^{th} \text{ entry of } A^k = \text{ strength of connectedness of } v_i - v_j \text{ walks of length } k. \tag{5}$$

Proof. Let $G = (V, E)$ be an IFG and let $A = \{ \langle \mu_{ij}, \nu_{ij} \rangle \}$ be the index matrix of G and the vertex set $V = \{v_1, v_2, \dots, v_n\}$. Let $A^k = \{ \langle b_{\mu_{ij}}, b_{\nu_{ij}} \rangle \}$ be the k^{th} power of the IF matrix A . Let us prove the Equation (1) by mathematical induction method on the power of A .

Initial Step: Let $k = 1$, then $A^k = A$. Then $A = \{ \langle \mu_{ij}, \nu_{ij} \rangle \}$, where μ_{ij} is μ - strength of connectedness of (v_i, v_j) walk of length 1 and ν_{ij} is ν - strength of connectedness of (v_i, v_j) walk of length 1 = $(CONN_{\mu(G)}(v_i, v_j), CONN_{\nu(G)}(v_i, v_j))$, where (v_i, v_j) is $v_i - v_j$ walk of length 1.

Inductive Step: Assume that the result is true for k . By the inductive hypothesis, $(i, j)^{th}$ entry of $A^k = (CONN_{\mu(G)}(v_i, v_j), CONN_{\nu(G)}(v_i, v_j))$, where (v_i, v_j) is $v_i - v_j$ walk of length k . Next we have to prove the result for $k + 1$.

$$(i, j)^{th} \text{ entry of } A^{k+1} = (i, j)^{th} \text{ entry}$$

of A^k $\bullet_{\max-\min}$ $(i, j)^{th}$ entry of $A = (\bigvee_{l=1}^n (a_{\mu_{il}} \wedge b_{\nu_{lj}}), \bigwedge_{l=1}^n (a_{\nu_{il}} \vee b_{\mu_{lj}})) = (\max(\min(\text{CONN}_{\mu(G)}(v_i, v_l), \text{CONN}_{\mu(G)}(v_l, v_j))), \min(\max(\text{CONN}_{\nu(G)}(v_i, v_l), \text{CONN}_{\nu(G)}(v_l, v_j))))$, where (v_i, v_l) is $v_i - v_l$ walk of length k and (v_l, v_j) is $v_l - v_j$ walk of length $1 = (\text{CONN}_{\mu(G)}(v_i, v_j), \text{CONN}_{\nu(G)}(v_i, v_j))$, where (v_i, v_j) is $v_i - v_j$ walks of length $k + 1, \forall v_l \in V$. Hence the result is true for every k . That is, $(i, j)^{th}$ entry of $A^k =$ strength of connectedness of $v_i - v_j$ walks of length k . ■

Theorem 3.9 Let $G = (V, E)$ be an intuitionistic fuzzy graph, where $V = \{v_1, v_2, \dots, v_n\}$ be the vertices of G . Let $A = \{< \mu_{ij}, \nu_{ij} >\}$ be the index matrix of G . Then $(\text{CONN}_{\mu(G)}(v_i, v_j), \text{CONN}_{\nu(G)}(v_i, v_j)) = (i, j)^{th}$ entry of $A + A^2 + \dots + A^{n-1}, \forall v_i \neq v_j \in V$ and $(\text{CONN}_{\mu(G)}(v_i, v_i), \text{CONN}_{\nu(G)}(v_i, v_i)) = (i, i)^{th}$ entry of $A + A^2 + \dots + A^n, \forall v_i \in V$

Proof. Let G be an IFG, where $V = \{v_1, v_2, \dots, v_n\}$ be the vertices of G . Let A be the index matrix of G and A^k be the power of IF matrix A . By Theorem 3.8, $(i, j)^{th}$ entry of $A^k = (\text{CONN}_{\mu(G)}(v_i, v_j), \text{CONN}_{\nu(G)}(v_i, v_j))$, where (v_i, v_j) is $v_i - v_j$ walks of length k .

Case(i): Let $v_i \neq v_j$. By Theorem 2.2, $v_i - v_j$ walk of length k contains $v_i - v_j$ path of length k . Since the vertex set V has $n-$ vertices, the $v_i - v_j$ path passes through at most $n-$ vertices. Therefore (v_i, v_j) is path of length less than or equal to $n - 1$

Hence,

$$\begin{aligned} (i, j)^{th} \text{ entry of } A + A^2 + \dots + A^{n-1} &= (\max(\text{CONN}_{\mu(G)}(v_i, v_j)), \min(\text{CONN}_{\nu(G)}(v_i, v_j))) \\ &= (\max(\max(S_{\mu(G)}(v_i, v_j)), \min(\min(S_{\nu(G)}(v_i, v_j)))) \\ &= (\text{CONN}_{\mu(G)}(v_i, v_j), \text{CONN}_{\nu(G)}(v_i, v_j)) \end{aligned}$$

where (v_i, v_j) is $v_i - v_j$ path of length less than or equal to $n - 1$

Therefore, $(\text{CONN}_{\mu(G)}(v_i, v_j), \text{CONN}_{\nu(G)}(v_i, v_j)) = (i, j)^{th}$ entry of $A + A^2 + \dots + A^{n-1}, \forall v_i \neq v_j \in V$.

Case(ii): Let $v_i = v_j \in V$. Since the vertex set V has $n-$ vertices, the closed $v_i - v_i$ path passes through at most $n-$ vertices. Therefore (v_i, v_i) is $v_i - v_i$ path of length less than or equal to n .

Hence,

$$\begin{aligned} (i, i)^{th} \text{ entry of } A + A^2 + \dots + A^n &= (\max(\text{CONN}_{\mu(G)}(v_i, v_i)), \min(\text{CONN}_{\nu(G)}(v_i, v_i))) \\ &= (\max(\max(S_{\mu(G)}(v_i, v_i)), \min(\min(S_{\nu(G)}(v_i, v_i)))) \\ &= (\text{CONN}_{\mu(G)}(v_i, v_i), \text{CONN}_{\nu(G)}(v_i, v_i)) \end{aligned}$$

where (v_i, v_i) is $v_i - v_j$ paths of length less than or equal to n

$$(\text{CONN}_{\mu(G)}(v_i, v_i), \text{CONN}_{\nu(G)}(v_i, v_i)) = (i, i)^{th} \text{ entry of } A + A^2 + \dots + A^n, \forall v_i \in V. \quad \blacksquare$$

Theorem 3.10 Let $G = (V, E)$ be a strong intuitionistic fuzzy graph and A be the index matrix of G . Let $C_k = \{< c_{\mu_{ij}}, c_{\nu_{ij}} >\} = A + A^2 + \dots + A^k$ and $C_{k-1} = \{< c'_{\mu_{ij}}, c'_{\nu_{ij}} >\} = A + A^2 + \dots + A^{k-1}$. Then G is connected and G^k is complete if and only if

$$\left\{ \begin{array}{l} \{< c_{\mu_{ij}}, c_{\nu_{ij}} >\} \neq < 0, 1 >, \text{ for every } i \text{ and } j \\ \{< c'_{\mu_{ij}}, c'_{\nu_{ij}} >\} = < 0, 1 >, \text{ for some } i \text{ and } j \end{array} \right.$$

Proof. Let $G = (V, E)$ be a strong intuitionistic fuzzy graph and A be the index matrix of G . Let A^k be the k^{th} power of the IF matrix A . Let G^k be a complete intuitionistic fuzzy graph. Then by Theorem 3.8:

$$\begin{aligned}
 &(i, j)^{th} \text{ entry of } A = (CONN_{\mu(G)}(v_i, v_j), CONN_{\nu(G)}(v_i, v_j)), \text{ where } (v_i, v_j) \text{ is} \\
 &v_i - v_j \text{ walk of length } 1 \\
 &(i, j)^{th} \text{ entry of } A^2 = (CONN_{\mu(G)}(v_i, v_j), CONN_{\nu(G)}(v_i, v_j)), \text{ where } (v_i, v_j) \\
 &\text{is } v_i - v_j \text{ walk of length } 2 \\
 &\dots\dots \\
 &(i, j)^{th} \text{ entry of } A^{k-1} = (CONN_{\mu(G)}(v_i, v_j), CONN_{\nu(G)}(v_i, v_j)), \text{ where } (v_i, v_j) \\
 &\text{is } v_i - v_j \text{ walk of length } k - 1 \\
 &(i, j)^{th} \text{ entry of } A^k = (CONN_{\mu(G)}(v_i, v_j), CONN_{\nu(G)}(v_i, v_j)), \text{ where } (v_i, v_j) \\
 &\text{is } v_i - v_j \text{ walk of length } k
 \end{aligned}$$

Therefore,

$$(i, j)^{th} \text{ entry of } A + A^2 + \dots + A^{k-1} = (\max(CONN_{\mu(G)}(v_i, v_j)), \min(CONN_{\nu(G)}(v_i, v_j))),$$

$$(i, j)^{th} \text{ entry of } A + A^2 + \dots + A^{k-1} = (CONN_{\mu(G)}(v_i, v_j), CONN_{\nu(G)}(v_i, v_j)) \tag{6}$$

where (v_i, v_j) is $v_i - v_j$ walk of length less than or equal to $k - 1$ and

$$(i, j)^{th} \text{ entry of } A + A^2 + \dots + A^k = (\max(CONN_{\mu(G)}(v_i, v_j)), \min(CONN_{\nu(G)}(v_i, v_j))),$$

$$(i, j)^{th} \text{ entry of } A + A^2 + \dots + A^k = (CONN_{\mu(G)}(v_i, v_j), CONN_{\nu(G)}(v_i, v_j)) \tag{7}$$

where (v_i, v_j) is $v_i - v_j$ walk of length $\leq k$.

From Equation (2), $(i, j)^{th}$ entry of $A + A^2 + \dots + A^{k-1}$ is the strength of connectedness of (v_i, v_j) walk of length $1, 2, \dots, k - 1$ except k .

Therefore, $\{< c'_{\mu_{ij}}, c'_{\nu_{ij}} >\} =$

$$\begin{cases} (CONN_{\mu(G)}(v_i, v_j), CONN_{\nu(G)}(v_i, v_j)) \neq < 0, 1 >, (v_i, v_j) \text{ is walk of length } 1, 2, \dots, k - 1 \\ < 0, 1 >, (v_i, v_j) \text{ is walk of length } k \end{cases} \tag{8}$$

Since G^k is complete, the maximum shortest path of length in G^k is k . Then there exist at least one path of length k in G^k . Therefore from Equation (6), (7) and (8),

$$(i, j)^{th} \text{ entry of } A + A^2 + \dots + A^k = (CONN_{\mu(G)}(v_i, v_j), CONN_{\nu(G)}(v_i, v_j)) \neq < 0, 1 >, \tag{9}$$

where (v_i, v_j) is walk of length k , $\forall i, j$.

Hence from Equation (8) and (9),

$$\begin{cases} \{< c_{\mu_{ij}}, c_{\nu_{ij}} >\} \neq < 0, 1 >, \text{ for every } i \text{ and } j \\ \{< c'_{\mu_{ij}}, c'_{\nu_{ij}} >\} = < 0, 1 >, \text{ for some } i \text{ and } j \end{cases} \tag{10}$$

Conversely, suppose that Equation (10) is true, then for each distinct pair i, j we have $C_k \neq 0$. Therefore, there exist at least one walk of length less than n from v_i to v_j . This implies that v_i is connected to v_j , for every $v_i, v_j \in V$. Hence G is connected. Again, let us assume that Equation (10) is true, then there exists $v_i - v_j$ walk of length $\leq k$ and shortest path of length k in G . By the Definition 1.38 and Theorem 2.2,

$$\mu_{ij}^k = \min(\mu_i, \mu_j) \text{ and } \nu_{ij}^k = \max(\nu_i, \nu_j), (v_i, v_j) \notin E \text{ and } d_{G^*}(v_i, v_j) \leq k \quad (11)$$

Since G is strong intuitionistic fuzzy graph and from the Equation(11),

$$\mu_{ij}^k = \min(\mu_i, \mu_j) \text{ and } \nu_{ij}^k = \max(\nu_i, \nu_j), \forall v_i, v_j \in V$$

Hence G^k is complete. ■

Corollary 3.11 Let $G = (V, E)$ be an intuitionistic fuzzy graph and A be the index matrix of G . Let $C_k = \{ \langle c_{\mu_{ij}}, c_{\nu_{ij}} \rangle \} = A + A^2 + \dots + A^k$ and $C_{k-1} = \{ \langle c'_{\mu_{ij}}, c'_{\nu_{ij}} \rangle \} = A + A^2 + \dots + A^{k-1}$. Then G is connected and $(G^*)^k$ complete if and only if

$$\left\{ \begin{array}{l} \{ \langle c_{\mu_{ij}}, c_{\nu_{ij}} \rangle \} \neq \langle 0, 1 \rangle, \text{ for every } i \text{ and } j \\ \{ \langle c'_{\mu_{ij}}, c'_{\nu_{ij}} \rangle \} = \langle 0, 1 \rangle, \text{ for some } i \text{ and } j \end{array} \right.$$

Corollary 3.12 Let $G = (V, E)$ be a strong directed intuitionistic fuzzy graph and A be the index matrix of G . Let $C_k = \{ \langle c_{\mu_{ij}}, c_{\nu_{ij}} \rangle \} = A + A^2 + \dots + A^k$ and $C_{k-1} = \{ \langle c'_{\mu_{ij}}, c'_{\nu_{ij}} \rangle \} = A + A^2 + \dots + A^{k-1}$. Then G is connected and G^k is complete if and only if

$$\left\{ \begin{array}{l} \{ \langle c_{\mu_{ij}}, c_{\nu_{ij}} \rangle \} \neq \langle 0, 1 \rangle, \text{ for every } i \text{ and } j \\ \{ \langle c'_{\mu_{ij}}, c'_{\nu_{ij}} \rangle \} = \langle 0, 1 \rangle, \text{ for some } i \text{ and } j \end{array} \right.$$

Theorem 3.13 Let G and H be two intuitionistic fuzzy graphs. Then G is co-weak isomorphic with H then G^k is homomorphic with H^k .

Proof. Proof follows from the definition 1.23, 1.25 and 1.38 . ■

Theorem 3.14 Let $G = (V, E)$ be an intuitionistic fuzzy graph. Let $A = \{ \langle \mu_{ij}, \nu_{ij} \rangle \}_{m \times m}$ and $B = \{ \langle b_{\mu_{ij}}, b_{\nu_{ij}} \rangle \}_{m \times n}$ be the index matrix and incidence matrix of G respectively. Then the entries of $B \bullet_{\max - \min} B^T$ are

$$\{ \langle b_{\mu_{ij}}, b_{\nu_{ij}} \rangle \}_{m \times m} = \left\{ \begin{array}{l} (\mu_{ij}, \nu_{ij}), \text{ if } i \neq j \\ (\max(\mu_{ik}), \min(\nu_{ik})), \text{ if } i = j, \forall v_k \in N_G(v_i) \end{array} \right.$$

Proof. Let $G = (V, E)$ be an intuitionistic fuzzy graph. Let $A = \{ \langle \mu_{ij}, \nu_{ij} \rangle \}$ and $B = \{ \langle b_{\mu_{ij}}, b_{\nu_{ij}} \rangle \}$ be the index matrix and incidence matrix of G of order $m \times m$ and $m \times n$ respectively. Let $B^T = \{ \langle b'_{\mu_{ij}}, b'_{\nu_{ij}} \rangle \}_{n \times m}$ be the transpose of the matrix B . Then the order of $B \bullet_{\max - \min} B^T$ is $m \times m$.

Case(i): Let $e_k \in (v_i, v_j), i \neq j$. By the Definition 1.35, the entries in B are as follows:

$$\{ \langle b_{\mu_{ij}}, b_{\nu_{ij}} \rangle \} = \begin{cases} (\mu_{ij}, \nu_{ij}), \forall e_k \in (v_i, v_j) \\ (0, 1), \text{ otherwise} \end{cases}$$

The $(i, j)^{th}$ entries of $B \bullet_{\max - \min} B^T$ is given by

$$(B \bullet_{\max - \min} B^T)_{ij} = (\vee_k (b_{\mu_{ik}} \wedge b'_{\mu_{kj}}), \vee_k (b_{\nu_{ik}} \wedge b'_{\nu_{kj}}))$$

That is,

$$(B \bullet_{\max - \min} B^T)_{ij} = (\mu_{ij}, \nu_{ij}), \text{ since } e_k \in (v_i, v_j) \tag{12}$$

Case(ii): Let $(v_i, v_j) \notin E$ of G .

Subcase(i): Let $v_i \in V$ is incident on $e_k \in E$ and $v_j \in V$ is not incident on $e_k \in E$, then $(b_{\mu_{ik}}, b_{\nu_{ik}}) \neq (0, 1) = (\mu(e_k), \nu(e_k))$, $(b_{\mu_{jk}}, b_{\nu_{jk}}) = (0, 1)$ and $(b'_{\mu_{kj}}, b'_{\nu_{kj}}) = (0, 1)$. Therefore,

$$(B \bullet_{\max - \min} B^T)_{ij} = (\vee_k (b_{\mu_{ik}} \wedge b'_{\mu_{kj}}), \wedge_k (b_{\nu_{ik}} \vee b'_{\nu_{kj}})) = (0, 1) \tag{13}$$

Subcase (ii): Let $v_i \in V$ is not incident on $e_k \in E$ and $v_j \in V$ is incident on $e_k \in E$, then $(b_{\mu_{ik}}, b_{\nu_{ik}}) = (0, 1)$ and $(b_{\mu_{jk}}, b_{\nu_{jk}}) \neq (0, 1) = \mu(e_k)$ and $(b'_{\mu_{kj}}, b'_{\nu_{kj}}) \neq (0, 1) = \mu(e_k)$. Therefore,

$$(B \bullet_{\max - \min} B^T)_{ij} = (\vee_k (b_{\mu_{ik}} \wedge b'_{\mu_{kj}}), \wedge_k (b_{\nu_{ik}} \vee b'_{\nu_{kj}})) = (0, 1) \tag{14}$$

Subcase(iii): Let $v_i, v_j \in V$ is not incident on $e_k \in E$, then $(b_{\mu_{ik}}, b_{\nu_{ik}}) = (0, 1)$ and $(b_{\mu_{jk}}, b_{\nu_{jk}}) = (0, 1)$ and $(b'_{\mu_{kj}}, b'_{\nu_{kj}}) = (0, 1)$. Therefore, $(B \bullet_{\max - \min} B^T)_{ij} = (\vee_k (b_{\mu_{ik}} \wedge b'_{\mu_{kj}}), \wedge_k (b_{\nu_{ik}} \vee b'_{\nu_{kj}})) = (0, 1)$

Hence from the above three cases,

$$(B \bullet_{\max - \min} B^T)_{ij} = (\vee_k (b_{\mu_{ik}} \wedge b'_{\mu_{kj}}), \vee_k (b_{\nu_{ik}} \wedge b'_{\nu_{kj}})) = (0, 1), \forall (v_i, v_j) \notin E \tag{15}$$

Case(iii): Let $v_i = v_j \in V$ in G .

$$\begin{aligned} (B \bullet_{\max - \min} B^T)_{ij} &= (\vee_k (b_{\mu_{ik}} \wedge b'_{\mu_{ki}}), \wedge_k (b_{\nu_{ik}} \vee b'_{\nu_{ki}})) \\ &= (\vee_k (b_{\mu_{ik}}), \wedge_k (b_{\nu_{ik}})) \\ &= ((b_{\mu_{i1}} \vee b_{\mu_{i2}} \vee \dots \vee b_{\mu_{in}}), (b_{\nu_{i1}} \wedge b_{\nu_{i2}} \wedge \dots \wedge b_{\nu_{in}})) \\ &= (\vee_k \mu_{ik}, \wedge_k \nu_{ik}), \forall e_k \in E \text{ is incident on } v_i \in V \end{aligned}$$

Hence from Equation (8), (9) and (10) and Case (iii),

$$\{ \langle b_{\mu_{ij}}, b_{\nu_{ij}} \rangle \}_{m \times m} = \begin{cases} \{ \langle \mu_{ij}, \nu_{ij} \rangle \}, \text{ if } i \neq j \\ (\max(\mu_{ik}), \min(\nu_{ik})), \text{ if } i = j, \forall v_k \in N_G(v_i) \end{cases}$$

■

Theorem 3.15 Let $G = (V, E)$ be an intuitionistic fuzzy graph and $G_L = (V_L, E_L)$ be a line intuitionistic fuzzy graph. Let $A = \{ \langle \mu_{ij}, \nu_{ij} \rangle \}$ and $B = \{ \langle b_{\mu_{ij}}, b_{\nu_{ij}} \rangle \}$ be the index matrix and incidence matrix of G respectively. Then the entries of $B^T \bullet_{\max - \min} B$ are

$$\{ \langle b_{\mu_{ij}}, b_{\nu_{ij}} \rangle \}_{n \times n} \begin{cases} (\mu_L(v_i, v_j), \nu_L(v_i, v_j)), \text{ if } i \neq j \\ (\max(\mu_{ik}), \min(\nu_{ik})), \text{ if } i = j, \forall v_k \in N_G(v_i) \end{cases}$$

where $(\mu_L(v_i, v_j), \nu_L(v_i, v_j))$ is the membership and non membership value of an edge $e_{ij} \in E_L$.

Proof. Let $G = (V, E)$ be an intuitionistic fuzzy graph. Let $A = \{ \langle \mu_{ij}, \nu_{ij} \rangle \}$ and $B = \{ \langle b_{\mu_{ij}}, b_{\nu_{ij}} \rangle \}$ be the index matrix and incidence matrix of G of order $m \times m$ and $m \times n$ respectively. Let $B^T = \{ \langle b'_{\mu_{ij}}, b'_{\nu_{ij}} \rangle \}_{n \times m}$ be the transpose of the matrix B . Then the order of $B^T \bullet_{\max - \min} B$ is $n \times n$.

Case(i): Let $e_i \in E, e_j \in E$ are incident on $v_k \in V$. Then $((b_{\mu_{ki}}, b_{\nu_{ki}}) \neq (0, 1) = (\mu(e_i), \nu(e_i)) = (b'_{\mu_{ik}}, b'_{\nu_{ik}}), (b_{\mu_{kj}}, b_{\nu_{kj}}) \neq (0, 1) = (\mu(e_j), \nu(e_j)) = (b'_{\mu_{jk}}, b'_{\nu_{jk}})$. The $(i, j)^{th}$ entry of $B^T \bullet_{\max - \min} B$ are as follows:

$$\begin{aligned} (B \bullet_{\max - \min} B^T)_{ij} &= (\vee_k (b'_{\mu_{ik}} \wedge b_{\mu_{kj}}, \wedge_k (b'_{\nu_{ik}} \vee b_{\nu_{kj}})) \\ &= (\vee (\mu(e_i) \wedge \mu(e_j)), \wedge (\nu(e_i) \vee \nu(e_j))) \\ &= ((\mu(e_i) \wedge \mu(e_j)), (\nu(e_i) \vee \nu(e_j))) \\ &= ((\mu_L(v_i) \wedge \mu_L(v_j)), (\nu_L(v_i) \vee \nu_L(v_j))), \text{ since by Definition 1.36} \\ &= (\mu_L(v_i, v_j), \nu_L(v_i, v_j)), (v_i, v_j) \in E \end{aligned}$$

Case(ii): Let $e_i \in E, e_j \in E$ are not incident on $v_k \in V$ Then $((b_{\mu_{ki}}, b_{\nu_{ki}}) = (0, 1) = (b'_{\mu_{ik}}, b'_{\nu_{ik}}), (b_{\mu_{kj}}, b_{\nu_{kj}}) = (0, 1) = (b'_{\mu_{jk}}, b'_{\nu_{jk}})$. The $(i, j)^{th}$ entry of $B^T \bullet_{\max - \min} B$ are as follows:

$$\begin{aligned} (B^T \bullet_{\max - \min} B)_{ij} &= (\vee_k (b'_{\mu_{ik}} \wedge b_{\mu_{ki}}, \wedge_k (b'_{\nu_{ik}} \vee b_{\nu_{ki}})) \\ &= (0, 1) \end{aligned}$$

Case(iii): Let $v_i = v_j \in V$ in G .

$$\begin{aligned} (B^T \bullet_{\max - \min} B)_{ij} &= (\vee_k (b'_{\mu_{ik}} \wedge b_{\mu_{ki}}, \wedge_k (b'_{\nu_{ik}} \vee b_{\nu_{ki}})) \\ &= (\vee_k (b_{\mu_{ik}}, \wedge_k (b_{\nu_{ik}})) \\ &= ((b_{\mu_{i1}} \vee b_{\mu_{i2}} \vee \dots \vee b_{\mu_{in}}), (b_{\nu_{i1}} \wedge b_{\nu_{i2}} \wedge \dots \wedge b_{\nu_{in}})) \\ &= (\vee_k \mu_{ik}, \wedge_k \nu_{ik}), \forall e_k \in E \text{ is incident with } v_i \in V \end{aligned}$$

Hence from the Cases (i) and (ii) and by Definition 1.36, $(B^T \bullet_{\max - \min} B)_{ij} = (\mu_L(v_i, v_j), \nu_L(v_i, v_j))$, if $i \neq j$

Hence from the Case (iii), $(B^T \bullet_{\max - \min} B)_{ij} = (\max(\mu_{ik}), \min(\nu_{ik}))$, if $i = j, \forall v_k \in N_G(v_i)$. ■

IV Conclusion

In this paper, we discussed the properties of the power of an intuitionistic fuzzy graph, subdivision intuitionistic fuzzy graph and line intuitionistic fuzzy graph. Intuitionistic fuzzy graph effectively expresses the approximate and interpolative reasoning used by humans when they employ linguistic propositions for deductive reasoning. The authors further extend this work so it can have application in decision making and network analysis.

References

- [1] Akram, M. and Davvaz, B., *Strong intuitionistic fuzzy graphs*, FILOMAT 26(1)(2012) 177-196.
- [2] Akram, M., Al-Shehrie N. O., *Intuitionistic fuzzy cycles and Intuitionistic fuzzy trees*, The Scientific World Journal, Volume 2014 (2014), Article ID 305836, 11 pages.
- [3] K. Atanassov, *Intuitionistic Fuzzy Sets: Theory and Applications*, Springer Physica-Verlag, Berlin, 1999.
- [4] Atanassov K., *Index matrix representation of the intuitionistic fuzzy graphs*, Fifth Scientific Session of the Mathematical Foundations of Artificial Intelligence Seminar, Sofia, Oct. 5, 1994, Preprint MRL-MFAIS-10-94, 36-41.
- [5] Atanassov K., *On index matrix interpretations of intuitionistic fuzzy graphs*, Notes on Intuitionistic Fuzzy Sets, Vol. 8 (2002), No. 4, 73-78.
- [6] K. Atanassov, *Generalized index matrices*, Compt. Rend. de l'Academie Bulgare des Sciences, Vol.40, 1987, No 11, 15-18.
- [7] Atanassov, K., *On index matrices. Part 1*. Advanced Studies in Contemporary Mathematics, 20(2), (2010)291-302.
- [8] Bondy, J.A and Murthy, U.S.R. *Graph Theory with Applications*, American Elsevier Publishing Co., New York, 1976.
- [9] Karunambigai, M.G., and Kalaivani, O.K., *Self Centered and Self Median Intuitionistic Fuzzy Graphs*, Advances in Fuzzy sets and systems, Volume 14, Number 2, 101-132, 2013.
- [10] M.G. Karunambigai and R. Parvathi, *Intuitionistic Fuzzy Graphs*, Journal of Computational Intelligence, Theory and Applications, Vol. 20, 139-150, 2006.
- [11] M.G.Karunambigai, R.Parvathi and O.K.Kalaivani, *A Study on Atanassovs Intuitionistic Fuzzy Graphs*, Proceedings of the International Conference on Fuzzy Systems, FUZZ-IEEE- 2011, Taipei, Taiwan, (2011)157-167.
- [12] A. Kaufmann, *An introduction to theory of fuzzy sub sets*, vol. 1, Academic Press, New York, 1975.
- [13] A. Nagoor Gani and S. Shajitha Begum, *Degree, Order and Size in Intuitionistic Fuzzy Graphs*, International Journal of Algorithms, Computing and Mathematics, Vol 3, Number 3, 2010.
- [14] Parvathi, R., Karunambigai, M.G., and Atanassov, K., *Operations on Intuitionistic Fuzzy Graphs*, Proceedings of IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), 1396-1401, 2009.
- [15] M.Pal, S.Khan and A.K.Shyamal, *Intuitionistic fuzzy matrices*, Notes on Intuitionistic Fuzzy Sets, 8 (2) (2002) 51-62.
- [16] R.Parvathi S.Thilagavathi, M.G.Karunambigai and G.Thamizhendhi, *Index Matrix representation of Intuitionistic fuzzy graphs*, Notes on Intuitionistic Fuzzy Sets, Vol. 20, No. 2, 100108 (2014)
- [17] Rosenfeld, A., 1975. *Fuzzy Graphs*, In *Fuzzy Sets and their Applications to Cognitive and Decision Processes*, Zadeh. L.A., Fu, K.S., Shimura, M., Eds ; Academic Press, New York, 77-95.
- [18] M. Sarwar and Akram, M., *An algorithm for computing certain metrics in intuitionistic fuzzy graphs*, Journal of Intelligent and Fuzzy Systems, 2015.
- [19] A. Shannon, K. Atanassov, *A First Step to a Theory of the Intuitionistic Fuzzy Graphs*, Proc. of the First Workshop on Fuzzy Based Expert Systems (D. akov, Ed.), Sofia, 59-61, 1994.
- [20] L.A.Zadeh, *Fuzzy sets*, Information and control 8, 1965, 338-353.

AUTHORS

First Author -M.G.Karunambigai, M.Sc,M.Phil,PGDCA, Ph.D, Department of Mathematics, Sri Vasavi College, Erode-638016, Tamilnadu, India. (karunsvc@yahoo.in).

Second Author - O.K.Kalaivani, M.Sc,M.Phil,PGDCA, Department of Mathematics, Sri Vasavi College, Erode-638016, Tamilnadu,India. (kalaivani83@gmail.com).