ISSN 2250-3153

Some results On Semiderivations of Semiprime Semirings

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Abstract- Let S be a semiprime semiring. An additive mapping $f : S \to S$ is called a semi derivation if there exists a function $g : S \to S$ such that (i) $f(xy) = f(x)g(y) + xf(y) = f(x)y + g(x)f(y)$,
(ii) $f(g(x)) = g(f(x))$ hold for all $x, y \in S$. In this paper we try to generalize some properties of prime rings with derivations to semiprime semirings with semiderivations.

Index Terms- Semirings, Semiprime semirings, Derivation, Semi derivation,Commuting mapping.

I. INTRODUCTION

Let S be a semiprime semiring with center $Z(S)$. For any $x, y \in S$, $[x, y], (x, y)$ represents $xy - yx, xy + yx$ respectively. Also we make use of basic commutator identities

$$[xy, z] = [x, z]y + x[y, z], \quad [x, yz] = y[x, z] + [x, y]z, \quad (xy, z) = (x, z)y + x[y, z] = [x, z]y + x(y, z)$$

J.C.Chang [6] studied on semi derivations of prime rings. He obtained some results of derivations of prime rings into semiderivations. H.E.Bell and W.S.Martindale III [7] investigated the commutativity property of a prime ring by means of semiderivations. C.L.Chuang [8] studied on the structure of semiderivations in prime rings. He obtained some remarkable results in connection with the semiderivatons. J.Bergen and P.Grezesczuk [5] obtained the commutativity properties of semiprime rings with the help of semi derivations. A.Firat [3] generalized some results of prime rings with derivations to the prime rings with semiderivations. In this paper we generalize some results of prime rings with semiderivations to the semiprime semirings with semiderivations.

II. PRELIMINARIES

Definition 2.1

A **semiring** S is a nonempty set S equipped with two binary operations $+$ and $\bullet$ such that

1. $(S,+)$ is a commutative monoid with identity element 0
2. $(S,\bullet)$ is a monoid with identity element 1
3.Multiplication left and right distributes over addition.

Definition 2.2

A semiring S is said to be **prime** if $xsy = 0$ implies $x = 0 \text{ or } y = 0 \text{ for all } x, y \in S.$

Definition 2.3

A semiring S is said to be **semiprime** if $xsx = 0$ implies $x = 0 \text{ for all } x \in S.$

Definition 2.4

A semiring S is said to be 2- torsion free if $2x = 0$ implies $x = 0 \text{ for all } x \in S.$

ISSN 2250-3153

Definition 2.5

A mapping $f : S \to S$ is said to be **commuting** on S if $[f(x), x] = 0$,holds $for\ all\ x \in S$ and is said to be **centralizing** on S if $[f(x), x] \in Z(S)$ holds $for\ all\ x \in S$.

Definition 2.6

An additive mapping $d : S \to S$ is called a **derivation** if $d(xy) = d(x)y + xd(y)$ holds $for\ all\ x, y \in S$.

Definition 2.7

An additive mapping $f : S \to S$ is called a **semiderivation** associated with a function $g : S \to S$ if for all $x, y \in S$ $(i) f(xy) = f(x)g(y) + xf(y)$, $(ii) f(g(x)) = g(f(x))$.

If $g = I$, ie an identity mapping of S then all semiderivations associated with g are merely ordinary derivations. If g is any endomorphism of S, then semiderivations are of the form $f(x) = x - g(x)$.

Example2.8

Let $S_1\ and\ S_2$ be two semiprime semirings. Let $S = S_1 \oplus S_2$. Let $\alpha_1 : S_1 \to S_1$ be an additive map, $\alpha_2 : S_2 \to S_2$ be a left and right S_2 module which is not a derivation. Define $f : S \to S$ such that $f(x_1, x_2) = (0, \alpha_2(x_2))$ and $g : S \to S$ such that $g(x_1, x_2) = (\alpha_1(x_1), 0)$ for all $x_1 \in S_1, x_2 \in S_2$. Define addition and multiplication on S by $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$ and $(x_1, x_2) \bullet (y_1, y_2) = (x_1 \bullet y_1, x_2 \bullet y_2)$. Then it can easily be seen that f is a semiderivation of S (with associated map g) which is not a derivation.

III. RESULTS

Lemma 3.1

Let S be a semiprime semiring, $a \in S$ .If S admits a semiderivation f such that $af(x) = 0\ or\ f(x)a = 0\ for\ all\ x \in S\ then\ a = 0\ or\ f = 0.$

Proof:

By hypothesis $af(x) = 0\ \ for\ all\ x \in S$

Replacing x by xy $for\ all\ x, y \in S$

$af(xy) = 0\ \ for\ all\ x, y \in S$

$af(x)g(y) + axf(y) = 0\ for\ all\ x, y \in S$

$axf(y) = 0\ for\ all\ x, y \in S$

$asf(y) = 0\ for\ all\ y \in S$

Since S is prime $a = 0\ or\ f(y) = 0\ for\ all\ y \in S$. Hence $a = 0\ or\ f = 0$.

Similarly we can prove for $f(x)a = 0$

Lemma 3.2

Let S be a semiprime semiring, f a nonzero semiderivation of S associated with a function g(not necessarily surjective). Then g is a homomorphism of S.

Proof:

$For\ any\ x, y, z \in S$

$$f(x(y+z)) = f(x)g(y+z) + xf(y+z)$$
$$= f(x)g(y+z) + xf(y) + xf(z) \quad (1)$$

$$\begin{aligned} &\text{Also for any } x, y, z \in S \\ &f(x(y+z)) = f(xy+xz) \\ &\quad = f(xy) + f(xz) \\ &\quad = f(x)g(y) + xf(y) + f(x)g(z) + xf(z) \end{aligned} \quad (2)$$

Comparing (1) and (2)

$$g(y+z) = g(y) + g(z) \text{ for all } y, z \in S$$

$$\begin{aligned} &\text{Now for any } x, y, z \in S \\ &f((xy)z) = f(xy)g(z) + xyf(z) \\ &\quad = f(x)g(y)g(z) + xf(y)g(z) + xyf(z) \end{aligned} \quad (3)$$

$$\begin{aligned} &\text{Also,} \\ &f((xy)z) = f(x(yz)) \\ &\quad = f(x)g(yz) + xf(yz) \\ &\quad = f(x)g(yz) + xf(y)g(z) + xyf(z) \end{aligned} \quad (4)$$

Comparing (3) and (4)

$$g(yz) = g(y)g(z) \text{ for all } y, z \in S$$

Hence g is a homomorphism of S.

Lemma 3.3

Let S be a semiprime semiring, f a semiderivation of S such that $f(S) \subseteq Z$ then $f = 0$ or S is commutative.

Proof:

By hypothesis $f(xy) \in Z \text{ for all } x, y \in S$

$ie,\ f(x)g(y) + xf(y) \in Z \text{ for all } x, y \in S$

Commuting this term with x

$$\begin{aligned} 0 &= [f(x)g(y) + xf(y), x] \\ &= [f(x)g(y), x] + [xf(y), x] \\ &= f(x)[g(y), x] + [f(x), x]g(y) + x[f(y), x] + [x, x]f(y) \\ &= f(x)[g(y), x] \end{aligned}$$

Since $f(x) \in Z$ and g is a surjective function of S, we have $f(x)s[y, x] = 0 \text{ for all } x, y \in S.$ Since S is prime $f(x) = 0 \text{ for all } x \in S$ or $[y, x] = 0 \text{ for all } x, y \in S$. ie, $f = 0$ or S is commutative.

Lemma 3.4

Let S be 2- torsion free semiprime semiring, f a semiderivation of S such that $f^2(x) = 0 \text{ for all } x \in S$,then $f = 0$.

Proof:

By hypothesis $f^2(x) = 0 \text{ for all } x \in S$

Replace x by xy

$f^2(xy)=0$, $for\, all\, x,y\in S$

$$0= f(f(xy)), for\, all\, x,y\in S$$
$$= f(f(x)g(y)+xf(y)), for\, all\, x,y\in S$$
$$= f^2(x)g(g(y))+ f(x)f(g(y))+ f(x)g(f(y))+xf^2(y), for\, all\, x,y\in S$$
$$=2f(x)f(g(y)), for\, all\, x,y\in S$$

Since S is 2- torsion free and g is surjective we have $f(x)f(y)=0\ for\, all\, x,y\in S$

Replace y by yz

$$f(x)f(yz)=0\ for\, all\, x,y,z\in S$$
$$f(x)f(y)g(z)+ f(x)yf(z)=0\ for\, all\, x,y,z\in S$$
$$f(x)yf(z)=0\ for\, all\, x,y,z\in S$$
$$f(x)sf(z)=0\ for\, all\, x,z\in S$$

Since S is prime $f(x)=0\, or f(z)=0\, all\, x,z\in S$

In both the cases $f=0$.

Lemma 3.5

Let S be a 2-torsion free semiprime semiring and a is an element in S. If S admits a semiderivation f such that $[f(x),a]=0\ for\, all\, x\in S\, then\ f=0\, or\, a\in Z(S)$.

Proof:

By hypothesis $[f(x),a]=0\ for\, all\, x\in S$

Replace x by xy

$$[f(xy),a]=0\ \ for\, all\, x,y\in S$$
$$0=[f(x)g(y)+xf(y),a]$$
$$=[f(x)g(y),a]+[xf(y),a]$$
$$= f(x)[g(y),a]+[f(x),a]g(y)+x[f(y),a]+[x,a]f(y)$$
$$= f(x)[g(y),a]+[x,a]f(y)\ for\, all\, x,y\in S$$

Since g is surjective, $0= f(x)[y,a]+[x,a]f(y)\ for\, all\, x,y\in S$

Replace y by $f(y)$

$$0= f(x)[f(y),a]+[x,a]f^2(y)\ for\, all\, x,y\in S$$
$$=[x,a]f^2(y)\, for\, all\, x,y\in S$$

Replace x by xz

$$0\ =[xz,a]f^2(y)\, for\, all\, x,y,z\in S$$
$$=x[z,a]f^2(y)+[x,a]zf^2(y)\, for\, all\, x,y,z\in S$$
$$=[x,a]zf^2(y)\, for\, all\, x,y,z\in S$$
$$=[x,a]sf^2(y)\, for\, all\, x,y\in S$$

Since S is Prime,

$[x,a]=0\, or\; f^2(y)=0$

$[x,a]=0 \Rightarrow a\in Z(S)\, and\; f^2(y)=0 \Rightarrow f=0\, by\, lemma\; 3.4$

Theorem 3.6

Let S be a 2-torsion free semiprime semiring and f a semiderivation of S such that $[f(S), f(S)]=0$ then $f=0$ or S is commutative.

Proof:

By hypothesis, $[f(S), f(S)]=0$

$[f(xy), f(z)]=0\; for\, all\; x,y,z\in S$

$[f(x)g(y)+xf(y), f(z)]=0$

$[f(x)g(y), f(z)]+[xf(y), f(z)]=0$

$f(x)[g(y), f(z)]+[f(x), f(z)]g(y)+x[f(y), f(z)]+[x, f(z)]f(y)=0$

$f(x)[g(y), f(z)]+[x, f(z)]f(y)=0\; for\, all\; x,y,z\in S$

Since g is surjective $f(x)[y, f(z)]+[x, f(z)]f(y)=0\; for\, all\; x,y,z\in S$

Put $y=f(y)$

$f(x)[f(y), f(z)]+[x, f(z)]f^2(y)=0\; for\, all\; x,y,z\in S$

$[x, f(z)]f^2(y)=0\; for\, all\; x,y,z\in S$

$[xu, f(z)]f^2(y)=0\; for\, all\; x,y,z,u\in S$

$x[u, f(z)]f^2(y)+[x, f(z)]uf^2(y)=0\; for\, all\; x,y,z,u\in S$

$[x, f(z)]uf^2(y)=0\; for\, all\; x,y,z,u\in S$

$[x, f(z)]sf^2(y)=0\; for\, all\; x,y,z\in S$

Since S is prime $[x, f(z)]=0\, or\; f^2(y)=0$

By lemma 3.4 and 3.5 $f=0$ or S is commutative.

Theorem 3.7

Let S be a semiprime semiring, f a semiderivation of S such that $[f(x),x]=0\; for\, all\; x\in S$

Then $f=0$ or S is commutative.

Proof:

By hypothesis $[f(x),x]=0\; for\, all\; x\in S$

Linearizing,

$0=[f(x),y]+[f(y),x]$

Replacing y by yx

$0=[f(x),yx]+[f(yx),x]\; for\, all\; x,y\in S$

$=[f(x),y]x+y[f(x),x]+[f(y)x+g(y)f(x),x]$

$=[f(x),y]x+f(y)[x,x]+[f(y),x]x+g(y)[f(x),x]+[g(y),x]f(x)$

$=[g(y),x]f(x)$

Since g is surjective

ISSN 2250-3153

$0 = [y, x] f(x) \; for\, all\; x, y \in S$

$\text{Re}\, place\; y\, by\; yz$

$0 = [yz, x] f(x) \; for\, all\; x, y, z \in S$

$= y[z, x] f(x) + [y, x] z f(x) \; for\, all\; x, y, z \in S$

$= [y, x] z f(x) \; for\, all\; x, y, z \in S$

$= [y, x] s f(x) \; for\, all\; x, y \in S$

Since S is prime $[y, x] = 0 \; or \; f(x) = 0$

This means that S is commutative or $f = 0.$

Theorem 3.8

Let S be a semiprime semiring, f a nonzero semiderivation of S such that $f([x, y]) = 0 \; for\, all\; x, y \in S.$ Then S is commutative

Proof:

By hypothesis $f([x, y]) = 0 \; for\, all\; x, y \in S$

Replacing y by xy

$0 = f([x, xy]) \, for\, all\; x, y \in S$

$= f(x[x, y]) \; for\, all\; x, y \in S$

$= f(x) g([x, y]) + x f([x, y]) \; for\, all\; x, y \in S$

$= f(x) g([x, y]) \; for\, all\; x, y \in S$

$= f(x)[g(x), g(y)] \sin ce\; g\; is\; a\; \hom omorphism$

$= f(x)[x, y] \sin ce\; g\; is\; surjective$

$\text{Re}\, place\; y\, by\; yz$

$0 = f(x)[x, yz] \, for\, all\; x, y, z \in S$

$= f(x) y [x, z] \, for\, all\; x, y, z \in S$

$= f(x) s [x, z] \, for\, all\; x, z \in S$

Since S is prime ,

$f(x) = 0 \, or \, [x, z] = 0$

This implies that $f = 0$ or S is commutative

Since f is nonzero S is commutative.

Theorem 3.9

Let S be a semiprime semiring, f a nonzero semiderivation of S such that

$f([x, y]) = \pm [x, y] \; for\, all\; x, y \in S.$ Then S is commutative .

By hypothesis, $f[x, y] = \pm [x, y] \; for\, all\; x, y \in S$

Replacing y by xy

$f([x, xy]) = \pm [x, xy] \; for\, all\; x, y \in S$

$f(x[x, y]) = \pm x[x, y]$

$f(x) g([x, y]) + x f([x, y]) = \pm x[x, y]$

$f(x) g([x, y]) = 0$

By theorem 3.8 S is commutative.

Lemma 3.10

Let S be a 2- torsion free semiprime semiring and f is a semiderivation of S with $g:S\to S$ is an onto endomorphism. Let $a\in S$, If the mapping $x\to[af(x),x]$ is commuting on S for all $x\in S$ *then* $x\to af(x)$ is commuting on S.

Proof:

By hypothesis, $[[af(x),x],x]=0$ *for all* $x\in S$

Linearising

$[[af(x),x],y]+[[af(x),y],x]=0$ *for all* $x,y\in S$

Replacing y by yx

$[[af(x),x],yx]+[[af(x),yx],x]=0$ *for all* $x,y\in S$

$$\begin{aligned}0&=[[af(x),x],yx]+[[af(x),yx],x] \text{ for all } x,y\in S\\&=[[af(x),x],y]x+y[[af(x),x],x]+[[af(x),y]x,x]+[y[af(x),x],x]\\&=[[af(x),x],y]x+[af(x),y][x,x]+[[af(x),y],x]x+y[[af(x),x],x]+[y,x][af(x),x]\\&=[y,x][af(x),x] \text{ for all } x,y\in S\end{aligned}$$

Re *place y by zy*

$$\begin{aligned}0&=[zy,x][af(x),x] \text{ for all } x,y,z\in S\\&=z[y,x][af(x),x]+[z,x]y[af(x),x]\\&=[z,x]y[af(x),x] \text{ for all } x,y,z\in S\end{aligned}$$

In particular

$$\begin{aligned}0&=[af(x),x]y[af(x),x] \text{ for all } x,y\in S\\&=[af(x),x]s[af(x),x] \text{ for all } x\in S\end{aligned}$$

By semiprimeness of S , $[af(x),x]=0$ *for all* $x\in S$

Hence $x\to af(x)$ is commuting on S.

Theorem 3.11

Let S be a non commutative 2- torsion free semiprime semiring and f is a semiderivation of S with $g:S\to S$ is an onto endomorphism. If the mapping $x\to[af(x),x]$ is commuting on S for all $x,y\in S$ *then* $a=0$ *or* $f=0$.

Proof:

By hypothesis, $[[af(x),x],x]=0$ *for all* $x\in S$

Then by lemma 3.10 $[af(x),x]=0$ *for all* $x\in S$

Linearizing

$$[af(x),y]+[af(y),x]=0 \text{ for all } x,y\in S \quad (2)$$

Replacing y by yx

$$[af(x), yx] + [af(yx), x] = 0$$
$$[af(x), y]x + y[af(x), x] + [af(y)x + ag(y)f(x), x] = 0$$
$$[af(x), y]x + [af(y), x]x + af(y)[x, x] + ag(y)[f(x), x] + [ag(y), x]f(x) = 0$$
$$ag(y)[f(x), x] + [ag(y), x]f(x) = 0 \quad (3)$$

Replacing $g(y)$ by $ag(y)$
$$a^2 g(y)[f(x), x] + [a^2 g(y), x]f(x) = 0$$
$$a^2 g(y)[f(x), x] + a[ag(y), x]f(x) + [a, x]ag(y)f(x) = 0 \quad (4)$$

Multiplying (3) by a
$$a^2 g(y)[f(x), x] + a[ag(y), x]f(x) = 0 \quad (5)$$

Comparing (4) and (5)
$$[a, x]ag(y)f(x) = 0 \text{ for all } x, y \in S$$

$$[a, x]asf(x) = 0$$

Since S is prime $[a, x]a = 0$ or $f(x) = 0$ for all $x \in S$

Let $[a, x]a = 0$

Replacing x by xy
$$[a, xy]a = 0 \text{ for all } x, y \in S$$
$$[a, x]ya = 0 \text{ for all } x, y \in S$$
$$[a, x]sa = 0 \text{ for all } x \in S$$

Since S is prime $[a, x] = 0$ or $a = 0$ for all $x \in S$

Since S is non-commutative $a = 0$ for all $x \in S$

Hence $a = 0$ or $f = 0$.

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