Some results On Semiderivations of Semiprime Semirings

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Abstract- Let S be a semiprime semiring. An additive mapping \( f : S \rightarrow S \) is called a semi derivation if there exists a function \( g : S \rightarrow S \) such that (i) \( f(xy) = f(x)g(y) + xf(y) = f(x)y + g(x)f(y) \), (ii) \( f(g(x)) = g(f(x)) \) hold for all \( x, y \in S \). In this paper we try to generalize some properties of prime rings with derivations to semiprime semirings with semiderivations.

Index Terms- Semirings, Semiprime semirings, Derivation, Semi derivation, Commuting mapping.

I. INTRODUCTION

Let S be a semiprime semiring with center \( Z(S) \). For any \( x, y \in S \), \( (x, y) \) represents \( xy - yx, xy + yx \) respectively. Also we make use of basic commutator identities

\[ [xy, z] = [x, z]y + x[y, z], \quad [x, yz] = y[x, z] + [x, y]z \]


II. PRELIMINARIES

Definition 2.1

A semiring \( S \) is a nonempty set \( S \) equipped with two binary operations \( + \) and \( \cdot \) such that

1. \( (S,+) \) is a commutative monoid with identity element \( 0 \)
2. \( (S,\cdot) \) is a monoid with identity element \( 1 \)
3. Multiplication left and right distributes over addition.

Definition 2.2

A semiring \( S \) is said to be prime if \( xxy = 0 \) implies \( x = 0 \) or \( y = 0 \) for all \( x, y \in S \).

Definition 2.3

A semiring \( S \) is said to be semiprime if \( xxx = 0 \) implies \( x = 0 \) for all \( x \in S \).

Definition 2.4

A semiring \( S \) is said to be 2- torsion free if \( 2x = 0 \) implies \( x = 0 \) for all \( x \in S \).
Definition 2.5

A mapping \( f : S \rightarrow S \) is said to be \textbf{commuting} on \( S \) if \( [f(x), x] = 0 \) for all \( x \in S \), and is said to be \textbf{centralizing} on \( S \) if \( [f(x), x] \in Z(S) \) for all \( x \in S \).

Definition 2.6

An additive mapping \( d : S \rightarrow S \) is called a \textbf{derivation} if \( d(xy) = d(x)y + xd(y) \) holds for all \( x, y \in S \).

Definition 2.7

An additive mapping \( f : S \rightarrow S \) is called a \textbf{semiderivation} associated with a function \( g : S \rightarrow S \) if for all \( x, y \in S \), \( (i) f(xy) = f(x)g(y) + xf(y) \) and \( (ii) f(g(x)) = g(f(x)) \).

If \( g = I \), i.e., an identity mapping of \( S \), then all semiderivations associated with \( g \) are merely ordinary derivations. If \( g \) is any endomorphism of \( S \), then semiderivations are of the form \( f(x) = x - g(x) \).

Example 2.8

Let \( S_1 \) and \( S_2 \) be two semiprime semirings. Let \( S = S_1 \oplus S_2 \). Let \( \alpha_1 : S_1 \rightarrow S_1 \) be an additive map, \( \alpha_2 : S_2 \rightarrow S_2 \) be a left and right \( S_2 \) module which is not a derivation. Define \( f : S \rightarrow S \) such that \( f(x_1, x_2) = (0, \alpha_2(x_2)) \) and \( g : S \rightarrow S \) such that \( g(x_1, x_2) = (\alpha_1(x_1), 0) \) for all \( x_1 \in S_1, x_2 \in S_2 \). Define addition and multiplication on \( S \) by \( (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2) \) and \( (x_1, x_2) \cdot (y_1, y_2) = (x_1 \cdot y_1, x_2 \cdot y_2) \). Then it can easily be seen that \( f \) is a semiderivation of \( S \) (with associated map \( g \)) which is not a derivation.

III. RESULTS

Lemma 3.1

Let \( S \) be a semiprime semiring, \( a \in S \). If \( S \) admits a semiderivation \( f \) such that \( af(x) = 0 \) or \( f(x)a = 0 \) for all \( x \in S \) then \( a = 0 \) or \( f = 0 \).

Proof:

By hypothesis \( af(x) = 0 \) for all \( x \in S \).

Replacing \( x \) by \( xy \) for all \( x, y \in S \), we have \( af(xy) = 0 \) for all \( x, y \in S \).

Similarly, \( af(y) + axf(y) = 0 \) for all \( x, y \in S \).

Therefore, \( af(y) = 0 \) for all \( x, y \in S \).

Hence, \( a = 0 \) or \( f = 0 \).

Similarly, we can prove for \( f(x)a = 0 \).

Lemma 3.2

Let \( S \) be a semiprime semiring, \( f \) a nonzero semiderivation of \( S \) associated with a function \( g \) (not necessarily surjective). Then \( g \) is a homomorphism of \( S \).

Proof:

For any \( x, y, z \in S \),

\[
\begin{align*}
f(x(y + z)) & = f(x)g(y + z) + xf(y + z) \\
& = f(x)g(y + z) + xf(y) + xf(z)
\end{align*}
\]
Also for any \( x, y, z \in S \)
\[
f(x(y + z)) = f(xy + xz)
= f(xy) + f(xz)
= f(x)g(y) + xf(y) + f(x)g(z) + xf(z)
\] (2)

Comparing (1) and (2)
\[
g(y + z) = g(y) + g(z) \text{ for all } y, z \in S
\]

Now for any \( x, y, z \in S \)
\[
f((xy)z) = f(xy)g(z) + yxf(z)
= f(x)g(y)g(z) + xf(y)g(z) + yxf(z)
\] (3)

Also,
\[
f((xy)z) = f(x(yz))
= f(x)g(yz) + xzf(yz)
= f(x)g(yz) + xf(y)g(z) + yzf(z)
\] (4)

Comparing (3) and (4)
\[
g(yz) = g(y)g(z) \text{ for all } y, z \in S
\]
Hence \( g \) is a homomorphism of \( S \).

**Lemma 3.3**

Let \( S \) be a semiprime semiring, \( f \) a semiderivation of \( S \) such that \( f(S) \subseteq Z \) then \( f = 0 \) or \( S \) is commutative.

Proof:

By hypothesis \( f(xy) \in Z \) for all \( x, y \in S \)

ie, \( f(x)g(y) + xf(y) \in Z \) for all \( x, y \in S \)

Commuting this term with \( x \)
\[
0 = \left[ f(x)g(y) + xf(y), x \right]
= \left[ f(x)g(y), x \right] + \left[ xf(y), x \right]
= f(x)\left[ g(y), x \right] + \left[ f(x), x \right]g(y) + x\left[ f(y), x \right] + \left[ x, x \right]f(y)
= f(x)g(y), x]
\]

Since \( f(x) \in Z \) and \( g \) is a surjective function of \( S \), we have \( f(x)\left[ y, x \right] = 0 \text{ for all } x, y \in S \). Since \( S \) is prime \( f(x) = 0 \text{ for all } x \in S \) or \( \left[ y, x \right] = 0 \text{ for all } x, y \in S \), ie, \( f = 0 \) or \( S \) is commutative.

**Lemma 3.4**

Let \( S \) be 2- torsion free semiprime semiring, \( f \) a semiderivation of \( S \) such that \( f^2(x) = 0 \text{ for all } x \in S \), then \( f = 0 \).

Proof:

By hypothesis \( f^2(x) = 0 \text{ for all } x \in S \)

Replace \( x \) by \( xy \)
Since $S$ is 2-torsion free and $g$ is surjective we have $f(x)f(y) = 0$ for all $x, y \in S$

Replace $y$ by $yz$

$f(x)f(yz) = 0$ for all $x, y, z \in S$

$f(x)f(y)g(z) + f(x)f(y)f(z) = 0$ for all $x, y, z \in S$

$f(x)f(y)f(z) = 0$ for all $x, y, z \in S$

Since $S$ is prime $f(x) = 0$ or $f(z) = 0$ for all $x, z \in S$

In both the cases $f = 0$.

Lemma 3.5

Let $S$ be a 2-torsion free semiprime semiring and $a$ is an element in $S$. If $S$ admits a semiderivation $f$ such that $[f(x), a] = 0$ for all $x \in S$ then $f = 0$ or $a \in Z(S)$.

Proof:

By hypothesis $[f(x), a] = 0$ for all $x \in S$

Replace $x$ by $xy$

$[f(xy), a] = 0$ for all $x, y \in S$

$0 = [f(x)g(y) + xf(y), a]$

$= [f(x)g(y), a] + [xf(y), a]$

$= f(x)[g(y), a] + [f(x), a]g(y) + x[f(y), a] + [x, a]f(y)$

$= f(x)[g(y), a] + [x, a]f(y)$ for all $x, y \in S$

Since $g$ is surjective, $0 = f(x)[y, a] + [x, a]f(y)$ for all $x, y \in S$

Replace $y$ by $f(y)$

$0 = f(x)[f(y), a] + [x, a]f^2(y)$ for all $x, y \in S$

$= [x, a]f^2(y)$ for all $x, y \in S$

Replace $x$ by $xz$

$0 = [xz, a]f^2(y)$ for all $x, y, z \in S$

$= x[z, a]f^2(y) + [x, a]zf^2(y)$ for all $x, y, z \in S$

$= [x, a]zf^2(y)$ for all $x, y, z \in S$

$= [x, a]f^2(y)$ for all $x, y \in S$

Since $S$ is Prime,
\[ [x, a] = 0 \text{ or } f^2(y) = 0 \]
\[ [x, a] = 0 \implies a \in Z(S) \text{ and } f^2(y) = 0 \implies f = 0 \text{ by lemma 3.4} \]

**Theorem 3.6**

Let \( S \) be a 2-torsion free semiprime semiring and \( f \) a semiderivation of \( S \) such that \( [f(S), f(S)] = 0 \) then \( f = 0 \) or \( S \) is commutative.

**Proof:**

By hypothesis, \( [f(S), f(S)] = 0 \)
\( [f(xy), f(z)] = 0 \) for all \( x, y, z \in S \)
\( [f(x)g(y) + xf(y), f(z)] = 0 \)
\( [f(x)g(y), f(z)] + [xf(y), f(z)] = 0 \)
\( f(x)[g(y), f(z)] + [f(x), f(z)]g(y) + x[f(y), f(z)] + [x, f(z)]f(y) = 0 \)
\( f(x)[g(y), f(z)] + [x, f(z)]f(y) = 0 \) for all \( x, y, z \in S \)

Since \( g \) is surjective \( f(x)[y, f(z)] + [x, f(z)]f(y) = 0 \) for all \( x, y, z \in S \)

Put \( y = f(y) \)
\( f(x)[f(y), f(z)] + [x, f(z)]f^2(y) = 0 \) for all \( x, y, z \in S \)
\( [x, f(z)]f^2(y) = 0 \) for all \( x, y, z \in S \)
\( [xu, f(z)]f^2(y) = 0 \) for all \( x, y, z, u \in S \)
\( x[u, f(z)]f^2(y) + [x, f(z)]uf^2(y) = 0 \) for all \( x, y, z, u \in S \)
\( [x, f(z)]uf^2(y) = 0 \) for all \( x, y, z, u \in S \)
\( [x, f(z)]uf^2(y) = 0 \) for all \( x, y, z \in S \)

Since \( S \) is prime \( [x, f(z)] = 0 \) or \( f^2(y) = 0 \)

By lemma 3.4 and 3.5 \( f = 0 \) or \( S \) is commutative.

**Theorem 3.7**

Let \( S \) be a semiprime semiring, \( f \) a semiderivation of \( S \) such that \( [f(x), x] = 0 \) for all \( x \in S \)

Then \( f = 0 \) or \( S \) is commutative.

**Proof:**

By hypothesis \( [f(x), x] = 0 \) for all \( x \in S \)

Linearizing,
\( 0 = [f(x), y] + [f(y), x] \)

Replacing \( y \) by \( yx \)
\( 0 = [f(x), yx] + [f(yx), x] \) for all \( x, y \in S \)
\( = [f(x), y]x + y[f(x), x] + [f(y)x + g(y)f(x), x] \)
\( = [f(x), y]x + f(y)[x, x] + [f(y), x]x + g(y)[f(x), x] + [g(y), x]f(x) \)
\( = [g(y), x]f(x) \)

Since \( g \) is surjective
0 = [y, x]f(x) for all x, y ∈ S

Re place y by yz

0 = [yz, x]f(x) for all x, y, z ∈ S

= y[z, x]f(x) + [y, x]f(z) for all x, y, z ∈ S

= [y, x]f(x) for all x, y, z ∈ S

Since S is prime [y, x] = 0 or f(x) = 0

This means that S is commutative or f = 0.

**Theorem 3.8**

Let S be a semiprime semiring, f a nonzero semiderivation of S such that \( f([x, y]) = 0 \) for all x, y ∈ S. Then S is commutative.

**Proof:**

By hypothesis \( f([x, y]) = 0 \) for all x, y ∈ S

Replacing \( y \) by \( xy \)

0 = f([x, xy]) for all x, y ∈ S

= f(x[x, y]) for all x, y ∈ S

= f(x)g([x, y]) + xf([x, y]) for all x, y ∈ S

= f(x)g([x, y]) for all x, y ∈ S

= f(x)[g(x), g(y)] since g is a homomorphism

= f(x)[x, y] since g is surjective

Re place y by yz

0 = f([x, yz]) for all x, y, z ∈ S

= f(x)[x, z] for all x, y, z ∈ S

= f(x)s[x, z] for all x, z ∈ S

Since S is prime,

\( f(x) = 0 \) or \([x, z] = 0 \)

This implies that \( f = 0 \) or S is commutative.

Since f is nonzero S is commutative.

**Theorem 3.9**

Let S be a semiprime semiring, f a nonzero semiderivation of S such that \( f([x, y]) = \pm [x, y] \) for all x, y ∈ S. Then S is commutative.

By hypothesis, \( f([x, y]) = \pm [x, y] \) for all x, y ∈ S

Replacing \( y \) by \( xy \)

\( f([x, xy]) = \pm [x, xy] \) for all x, y ∈ S

\( f(x[x, y]) = \pm x[x, y] \)

\( f(x)g([x, y]) + xf([x, y]) = \pm x[x, y] \)

\( f(x)g([x, y]) = 0 \)
By theorem 3.8 S is commutative.

**Lemma 3.10**

Let S be a 2-torsion free semiprime semiring and f is a semiderivation of S with \( g : S \rightarrow S \) is an onto endomorphism. Let \( a \in S \). If the mapping \( x \rightarrow ([af(x)], x) \) is commuting on S for all \( x \in S \) then \( x \rightarrow af(x) \) is commuting on S.

**Proof:**

By hypothesis, \([af(x), x] = 0 \) for all \( x \in S \)
Linearising \([af(x), y] + [af(x), x] = 0 \) for all \( x, y \in S \)
Replacing \( y \) by \( xy \)
\([af(x), x] + [af(x), y] = 0 \) for all \( x, y \in S \)

\[0 = [af(x), x] + [af(x), y] \text{ for all } x, y \in S\]
\[= [af(x), x] + [af(x), y] + [af(x), x] + [af(x), y\] + [yaf(x), x] + [y, xaf(x), x]
\[= [y, xaf(x), x] \text{ for all } x, y \in S\]

Re place \( y \) by \( zy \)
\[0 = [zy, xaf(x), x] \text{ for all } x, y, z \in S\]
\[= [z, x][af(x), x] + [z, x][af(x), x]\]
\[= [z, x][af(x), x] \text{ for all } x, y, z \in S\]

In particular
\[0 = [af(x), x]y[af(x), x] \text{ for all } x, y \in S\]
\[=[af(x), x]y[af(x), x] \text{ for all } x \in S\]

By semiprimeness of S, \([af(x), x] = 0 \) for all \( x \in S \)
Hence \( x \rightarrow af(x) \) is commuting on S.

**Theorem 3.11**

Let S be a non commutative 2-torsion free semiprime semiring and f is a semiderivation of S with \( g : S \rightarrow S \) is an onto endomorphism. If the mapping \( x \rightarrow [af(x), x] \) is commuting on S for all \( x, y \in S \) then \( a = 0 \) or \( f = 0 \).

**Proof:**

By hypothesis, \([af(x), x] = 0 \) for all \( x \in S \)
Then by lemma 3.10 \([af(x), x] = 0 \) for all \( x \in S \)
Linearizing \([af(x), y] + [af(y), x] = 0 \) for all \( x, y \in S \)
Replacing \( y \) by \( yx \)
\[ \begin{align*}
[af(x), yx] + [af(yx), x] &= 0 \\
[af(x), y]x + y[af(x), x] + [af(y), x] + ag(y)f(x), x] &= 0 \\
[af(x), y]x + [af(y), x]x + [af(y), y]x + [af(x), f(x), x] + [ag(y), x]f(x) &= 0 \\
ag(y)[f(x), x] + [ag(y), x]f(x) &= 0
\end{align*} \]

(3)

Replacing \( g(y) \) by \( ag(y) \)
\[ \begin{align*}
a^2g(y)[f(x), x] + [a^2g(y), x]f(x) &= 0 \\
a^2g(y)[f(x), x] + a[ag(y), x]f(x) + [a, x]ag(y)f(x) &= 0
\end{align*} \]

(4)

Multiplying (3) by \( a \)
\[ \begin{align*}
a^2g(y)[f(x), x] + a[ag(y), x]f(x) &= 0
\end{align*} \]

(5)

Comparing (4) and (5)
\[ \begin{align*}
[a, x]ag(y)f(x) &= 0 \text{ for all } x, y \in S
\end{align*} \]

Since \( S \) is prime
\[ \begin{align*}
[a, x]a = 0 \text{ or } f(x) = 0 \text{ for all } x \in S
\end{align*} \]

Let \( [a, x]a = 0 \)

Replacing \( x \) by \( xy \)
\[ \begin{align*}
[a, xy]a &= 0 \text{ for all } x, y \in S \\
[a, x]ya &= 0 \text{ for all } x, y \in S \\
[a, x]a &= 0 \text{ for all } x \in S
\end{align*} \]

Since \( S \) is prime
\[ \begin{align*}
[a, x]a = 0 \text{ or } a = 0 \text{ for all } x \in S
\end{align*} \]

Since \( S \) is non-commutative
\[ \begin{align*}
a = 0 \text{ for all } x \in S
\end{align*} \]

Hence
\[ \begin{align*}
a = 0 \text{ or } f = 0.
\end{align*} \]

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