AQ-Functional Equation in Paranormed Spaces

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***Abstract*- In this paper, we introduce and investigate the general solution of a new AQ-functional equation**

$$2a^2\left[f\left(\frac{x+y}{a}\right)+f\left(\frac{x-y}{a}\right)\right]=(1+a)\left[f(x+y)+f(x-y)\right]+(1-a)\left[f(-x+y)+f(-x-y)\right]$$

where $a \neq 0, \pm 1$ and discuss its Hyers-Ulam stability in paranormed spaces.

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I. Introduction

Functional equations of various forms were dealt in the last three decades regressively by many authors [2, 5, 8, 9]. Ulam [13] raised a question concerning the stability of group homomorphism as follows :

Let G_1 be a group and let G_2 be a metric group with the metric d(.,.). Given ε >0, does there exist a δ >0 such that if a function $h:G_1 \to G_2$ satisfies the inequality $d\left(h(xy),h(x)h(y)\right)<\delta$ for all $x,y \in G_1$, then there exist a homomorphism $H:G_1 \to G_2$ with $d\left(h(x),H(x)\right)<\varepsilon$ for all $x \in G_1$?

When G_1 and G_2 are Banach spaces, D.H. Hyers [7]solved the above question for the case of approximately additive functions. Later Th.M.Rassias [10] given a generalized version of the theorem of Hyers for approximately linear mappings. Then many mathematicians like Z.Gajda [3] , R.Ger [2], P.Gavruta [4], S. Czerwik [1] and J.M.Rassias [8] contributed a lot for the development of stability theory for various forms of functional equations . the functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

is called quadratic functional equation because every solution of the quadratic functional equation is said to be a quadratic mapping. In the same way

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \qquad (1.1)$$

and

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y) \qquad (1.2)$$

are called cubic and quartic functional equations because $f(x)=x^3$ and $f(x)=x^4$ respectively satisfies the equations (1.1) and (1.2) . Recently J. M. Rassias and H.M.Kim [9] investigated Generalized Hyers-Ulam stability for general additive functional equations in quasi- β-normed spaces , M.E. Gordji and M.B. Savad kouhi [6] studied the stability properties of a mixed type additive, quadratic and cubic functional equation

$$f(x+3y)+f(x-3y)=9f(x+y)+9f(x-y)-16f(x) \qquad (1.3)$$

in random normed spaces. Very recently , C.Park and J.R.Lee [11] proved some results on the Hyers-Ulam stability of an additive-quadratic-cubic-quatraic functional equation

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)-6f(x)+f(2y) + f(-2y)-4f(y)-4f(-y) \qquad (1.4)$$

in paranormed spaces.

In this paper, authors are interested in finding the solutions and some results on Hyers-Ulam stability of a new Additive-Quadratic functional equation

$$2a^2\left[f\left(\frac{x+y}{a}\right)+f\left(\frac{x-y}{a}\right)\right]=(1+a)\left[f(x+y)+f(x-y)\right]+(1-a)\left[f(-x+y)+f(-x-y)\right]$$

where $a \neq 0, \pm 1$, in paranormed spaces. We consider some basic concepts concerning Frechet spaces and paranormed spaces.

Definition 1.1 [13]

Let X be a vector space. A Paranorm $P: X \to [0,\infty)$ is a function on X such that

(i) $P(0)=0$

(ii) $P(-x)=P(x)$

(iii) $P(x+y)\leq P(x)+P(y)$ (Triangle inequality)

(iv) If $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{t_n\}\subset X$ with

$P(x_n - x)\to 0$, then $P(t_n x_n - tx)\to 0$ (continuity of multiplication).

The pair (X,P) is called a Paranormed space if P is a paranorm on X. The paranorm is called total if, in addition, we have

(v) $P(x)=0 \Rightarrow x=0$

A Frechet space is a total and complete paranormed space.

In this paper, we first discuss the solution of a new additive and quadratic functional equation :

$$2a^2\left[f\left(\frac{x+y}{a}\right)+f\left(\frac{x-y}{a}\right)\right]=(1+a)\left[f(x+y)+f(x-y)\right] +(1-a)\left[f(-x+y)+f(-x-y)\right] \qquad (1.5)$$

with $a \neq 0,\pm 1$, then we investigated the Hyers-Ulam Stability of (1.5) in paranormed spaces

II. Solution of the Equation (1.5)

In this Section, let E_1 and E_2 denote real vectors spaces, we will prove the following two Lemmas, which will be useful to prove our main theorems.

Lemma 2.1

If $f: E_1 \to E_2$ is an even function, satisfies equation (1.5) for all $x, y \in E_1$. Then f is quadratic .

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Proof

Replacing (x,y) by $(0,0)$ in (1.5), we obtain

$$f(0)=0, \qquad \forall x \in E_1. \tag{2.1}$$

The function f is even and therefore $f(-x)=f(x)$ for all $x \in E_1$.
Equation (1.5) becomes ,

$$a^2\left[f\left(\frac{x+y}{a}\right)+f\left(\frac{x-y}{a}\right)\right]=f(x+y)+f(x-y), \quad \forall x,y \in E_1 \tag{2.2}$$

Replacing (x,y) by (z,z) in (2.2), we arrive that

$$a^2 f\left(\frac{2z}{a}\right)=f(2z), \qquad \forall z \in E_1 \tag{2.3}$$

Replacing z by $\frac{y}{2}$ in (2.3), we arrive that

$$a^2 f\left(\frac{y}{a}\right)=f(y), \qquad \forall y \in E_1 \tag{2.4}$$

Again, replacing y by ax in (2.4), we obtain

$$f(ax)=a^2 f(x), \qquad \forall x \in E_1 \tag{2.5}$$

Therefore $f: E_1 \to E_2$ is quadratic .

Lemma 2.2

If $f: E_1 \to E_2$ be an odd function, satisfies equation (1.5) for all $x,y \in E_1$. Then f is additive.

Proof

The function f is odd and therefore $f(-x)=-f(x)$ for all $x,y \in E_1$, Equation (1.5) becomes,

$$a\left[f\left(\frac{x+y}{a}\right)+f\left(\frac{x-y}{a}\right)\right]=f(x+y)+f(x-y), \qquad \forall x,y \in E_1 \tag{2.6}$$

Replacing (x,y) by (z,z) in (2.6), using equation (2.1) we arrive that

$$a\, f\left(\frac{2z}{a}\right)=f(2z), \qquad \forall z \in E_1 \tag{2.7}$$

Replacing z by $\frac{y}{2}$ in (2.7), we arrive that

$$a\, f\left(\frac{y}{a}\right) = f(y), \qquad \forall\, y \in E_1 \qquad (2.8)$$

Again, replacing y by ax in (2.8), we obtain

$$f(ax) = a\, f(x), \qquad \forall\, x \in E_1 \qquad (2.9)$$

Therefore $f : E_1 \to E_2$ is additive .

Theorem 2.3

A function $f : E_1 \to E_2$ satisfies equation (1.5) for all $x, y \in E_1$, if and only if there exists a symmetric bi-additive function $B : E_1 \times E_1 \to E_2$ and an additive function $A : E_1 \to E_2$ such that

$$f(x) = B(x,x) + A(x), \qquad \forall x \in E_1$$

Proof . Suppose there exists a symmetric bi-additive function $B : E_1 \times E_1 \to E_2$ and an additive function $A : E_1 \to E_2$ such that

$$f(x) = B(x,x) + A(x), \qquad \forall x \in E_1 \qquad (2.10)$$

then using (2.10), we obtain

$$f\left(\frac{x+y}{a}\right) = B\left(\frac{x+y}{a}, \frac{x-y}{a}\right) + A\left(\frac{x+y}{a}, \frac{x-y}{a}\right) \qquad (2.11)$$

$$f\left(\frac{x-y}{a}\right) = B\left(\frac{x-y}{a}, \frac{x-y}{a}\right) + A\left(\frac{x-y}{a}, \frac{x-y}{a}\right) \qquad (2.12)$$

for all $x, y \in E_1$ .From (2.11) and (2.12), we obtain

$$a^2\left[f\left(\frac{x+y}{a}\right) + f\left(\frac{x-y}{a}\right)\right] = a^2\left[B\left(\frac{x+y}{a}, \frac{x+y}{a}\right) + A\left(\frac{x+y}{a}, \frac{x+y}{a}\right)\right] + a^2\left[B\left(\frac{x-y}{a}, \frac{x-y}{a}\right) + A\left(\frac{x-y}{a}, \frac{x-y}{a}\right)\right] \qquad (2.13)$$

for all $x, y \in E_1$. Using properties of symmetric bi-additive function in (2.13), we arrive

$$2a^2\left[f\left(\frac{x+y}{a}\right) + f\left(\frac{x-y}{a}\right)\right] = (1+a)\left[f(x+y) + f(x-y)\right] + (1-a)\left[f(-x+y) + f(-x-y)\right], \qquad \forall x, y \in E_1 .$$

Hence the function satisfies (1.5).

Conversely, we decompose f into even part and the odd part by letting $f_e(x)=\frac{f(x)+f(-x)}{2}$ and $f_0(x)=\frac{f(x)-f(-x)}{2}$ $x\in E_1$. Replacing x by -x ,y by -y in (1.5) and adding, subtracting the resultant equation with (1.5), we find that $f_e(x)$, $f_o(x)$ $x,y\in E_1$ satisfies (1.5). Hence by Lemma 2.1 and Lemma 2.2, we obtain that the functions $f_e(x)$ and $f_o(x)$ are quadratic and additive respectively. It shows that there exists a symmetric bi-additive function $B:E_1\times E_1\to E_2$ such that $f_e(x)=B(x,x)$ and an additive function $A:E_1\to E_2$ such that $A(x)=f_o(x)$ and $f(x)=B(x,x)+A(x), \quad \forall x\in E_1$.

III. HYERS – ULAM STABILITY OF THE FUNCTIONAL EQUATION (1.5) AN ODD MAPPING CASE

For a given mapping f , we define

$$D f(x,y)=2a^2\left[f\left(\frac{x+y}{a}\right)+f\left(\frac{x-y}{a}\right)\right]-(1+a)\left[f(x+y)+f(x-y)\right]$$
$$-(1-a)\left[f(-x+y)+f(-x-y)\right]$$

In this Section and section (4), we assume that (E_1,P) is a Frechet space and $(E_2,\|.\|)$ is a Banach space.

Theorem 3.1

Let r and θ be positive real numbers with $r>1$, and let $f:E_2\to E_1$ be an odd mapping such that

$$P\left[D f(x,y)\right]\le\theta\left(\|x\|^r+\|y\|^r\right) \qquad (3.1)$$

for all $x,y\in E_2$. Then there exists a unique additive mapping $A:E_2\to E_1$ such that

$$P\left[f(x)-A(x)\right]\le\frac{\theta}{2^r}\left[\frac{a^{r-1}}{a^r-a}\right]\|x\|^r \qquad \forall x\in E_2 \qquad (3.2)$$

Proof . Using oddness of f in (3.1), we obtain

$$\left[2a^2\left(f\left(\frac{x+y}{a}\right)+f\left(\frac{x-y}{a}\right)\right)-2a\left(f(x+y)+f(x-y)\right)\right]\le\theta\left(\|x\|^r+\|y\|^r\right) \qquad (3.3)$$

for all $x,y\in E_2$. Replace (x,y) by (z,z) in (3.3), we obtain

$$P\left[2a^2 f\left(\frac{2z}{a}\right)-2af(2z)\right]\le\theta 2\|z\|^r \quad \forall z\in E_2 \qquad (3.4)$$

Replace z by $\frac{x}{2}$ in (3.4) , we obtain

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$$P\left[f(x)-af\left(\frac{x}{a}\right)\right]\leq\frac{\theta}{a2^r}\|x\|^r \qquad \forall x\in E_2 \tag{3.5}$$

Hence

$$P\left[a^u f\left(\frac{x}{a^u}\right)-a^v f\left(\frac{x}{a^v}\right)\right]\leq\sum_{j=u}^{v-1}\frac{\theta a^{j-1}}{a^{rj}2^r}\|x\|^r,\quad \forall x\in E_2 \tag{3.6}$$

For all non negative integers u and v with $v>u$ and all $x\in E_2$. It follows from (3.6) that the sequence $\left\{a^k f\left(\frac{x}{a^k}\right)\right\}$ is a Cauchy sequence for all $x\in E_2$, since E_1 is complete, the sequence $\left\{a^k f\left(\frac{x}{a^k}\right)\right\}$ converges for all $x\in E_2$. Now we define the mapping $A:E_2\to E_1$ by

$$A(x)=\lim_{k\to\infty}a^k f\left(\frac{x}{a^k}\right), \qquad \forall x\in E_2 .$$

by (3.1) , we get

$$P\left[DA(x,y)\right]=\lim_{k\to\infty}P\left[a^k Df\left(\frac{x}{a^k},\frac{y}{a^k}\right)\right]$$

$$\leq\lim_{k\to\infty}\frac{\theta}{a^{k(r+1)}}\left(\|x\|^r+\|y\|^r\right)=0, \qquad \forall x,y\in E_{2.} .$$

So $DA(x,y)=0$. Since $f:E_2\to E_1$ is odd , $A:E_2\to E_1$ is odd . So the mapping $A:E_2\to E_1$ is additive. Moreover, letting $u=0$ and passing the limit $v\to\infty$ in (3.6), we arrive (3.2).So there exists an additive mapping $A:E_2\to E_1$ satisfing (3.2).Now, let $A':E_2\to E_1$ be another additive mapping satisfing (3.2). Then we have

$$P\left[A(x)-A'(x)\right]=P\left[a^s A\left(\frac{x}{a^s}\right)-a^s A'\left(\frac{x}{a^s}\right)\right]$$

$$\leq P\left[a^s\left(A\left(\frac{x}{a^s}\right)-g\left(\frac{x}{a^s}\right)\right)\right]+P\left[a^s\left(A'\left(\frac{x}{a^s}\right)-g\left(\frac{x}{a^s}\right)\right)\right]$$

$$\leq\left(\frac{a^{r-1}}{2^{r-1}\left(a^r-a\right)}\right)\left(\frac{\theta}{a^{s(r-1)}}\right)\|x\|^r\to 0 \quad as \quad s\to\infty$$

for all $x\in E_2$. So we can conclude that $A(x)=A'(x), \quad \forall x\in E_2$. This proves the uniqueness of A . Thus the mapping $A:E_2\to E_1$ is a unique additive mapping satisfying (3.2).

Corollary 3.2

Let r and θ be positive real numbers with $r\geq 1$, and let $f:E_2\to E_1$ be an odd mapping such that

$$P\left[D\,f\left(x,y\right)\right]\leq\begin{Bmatrix}\theta\left(\|x\|^{r}\,\|y\|^{r}\right),\\ \theta\left(\|x\|^{2r}+\|y\|^{2r}+\|x\|^{r}\,\|y\|^{r}\right),\end{Bmatrix}$$

for all $x,y\in E_2$. Then there exists a unique additive mappings $A:E_2\to E_1$ satisfying

$$P\left[\,f\left(x\right)-A\left(x\right)\right]\leq\begin{Bmatrix}\lambda_1\,\|x\|^{2r}\,,\\ 3\lambda_1\,\|x\|^{2r}\,,\end{Bmatrix}$$

where $\lambda_1=\dfrac{\theta a^{2r-1}}{2^{2r+1}\left(a^{2r}-a\right)}$, for all $x\in E_2$.

Theorem 3.3

Let r be positive real numbers with $r<1$, and let $f:E_1\to E_2$ be an odd mapping such that

$$\left\|Df\left(x,y\right)\right\|\leq P\left(x\right)^{r}+P\left(y\right)^{r}\quad\forall x,y\in E_1. \tag{3.7}$$

Then there exists a unique additive mapping $A:E_1\to E_2$ such that

$$\left\|f\left(x\right)-A\left(x\right)\right\|\leq\frac{1}{a-a^{r}}\left(\frac{a}{2}\right)^{r}P\left(x\right)^{r},\quad\forall x\in E_1\quad. \tag{3.8}$$

Proof . Using oddness of f in (3.7), we obtain

$$\left\|2a^{2}\left(f\left(\frac{x+y}{a}\right)+f\left(\frac{x-y}{a}\right)\right)-2a\left(f\left(x+y\right)+f\left(x-y\right)\right)\right\|$$
$$\leq P\left(x\right)^{r}+P\left(y\right)^{r},\quad\forall x,y\in E_1 \tag{3.9}$$

Replace $\left(x,y\right)$ by $\left(z,z\right)$ in (3.9), we obtain

$$\left\|2a^{2}f\left(\frac{2z}{a}\right)-2a\,f\left(2z\right)\right\|\leq 2P\left(z\right)^{r},\quad\forall\,z\in E_1 \tag{3.10}$$

Replace z by $\dfrac{y}{2}$ in (3.10) , we obtain

$$\left\|a\,f\left(\frac{y}{a}\right)-f\left(y\right)\right\|\leq\frac{1}{a2^{r}}P\left(y\right)^{r}\quad,\quad\forall\,y\in E_1 \tag{3.11}$$

Again, replacing y by ax in (3.11), we obtain

$$\left\|f\left(x\right)-\frac{1}{a}f\left(ax\right)\right\|\leq\frac{a^{r}1}{a^{2}2^{r}}P\left(x\right)^{r}\qquad\forall\,x\in E_1 \tag{3.12}$$

Hence

$$\left\|\frac{1}{a^{u}}f\left(a^{u}x\right)-\frac{1}{a^{v}}f\left(a^{v}x\right)\right\|\leq\sum_{j=u}^{v-1}\frac{a^{r(j+1)}}{a^{j+2}2^{r}}\|x\|^{r},\quad\forall x\in E_1 \tag{3.13}$$

For all non negative integers u and v with $v>u$ and all $x\in E_1$. It follows from (3.12) that the sequence $\left\{\frac{1}{a^k} f\left(a^k x\right)\right\}$ is a Cauchy sequence for all $x\in E_1$, since E_2 is complete, the sequence $\left\{\frac{1}{a^k} f\left(a^k x\right)\right\}$ converges for all $x\in E_1$. Now we define the mapping $A: E_1 \to E_2$ by

$$A(x)=\lim_{k\to\infty}\frac{1}{a^k} f\left(a^k x\right), \qquad \forall x\in E_1 .$$

by (3.7) , we get

$$\|D A(x,y)\| =\lim_{k\to\infty}\left\|\frac{1}{a^k} D f\left(a^k x, a^k y\right)\right\|$$

$$\le \lim_{k\to\infty}\frac{1}{a^{k(1-r)}}\left(P(x)^r+P(y)^r\right)=0, \qquad \forall x,y\in E_{1.} .$$

So $D A(x,y)=0$. Since $f: E_1 \to E_2$ is odd , $A: E_1 \to E_2$ is odd . So the mapping $A: E_1 \to E_2$ is additive. Moreover, letting $u=0$ and passing the limit $v\to\infty$ in (3.12), we arrive (3.8).So there exists an additive mapping $A: E_1 \to E_2$ satisfing (3.8). Now, let $A^{'}: E_1 \to E_2$ be another additive mapping satisfing (3.8). Then we have

$$\left\|A(x)-A^{'}(x)\right\|=\left\|\frac{1}{a^s}A\left(a^s x\right)-\frac{1}{a^s}A^{'}\left(a^s x\right)\right\|$$

$$\le\left\|\frac{1}{a^s}\left(A\left(a^s x\right)-g\left(a^s x\right)\right)\right\|+\left\|\frac{1}{a^s}\left(A^{'}\left(a^s x\right)-g\left(a^s x\right)\right)\right\|$$

$$\le\left(\frac{a^r}{2^{r-1}\left(a^{s(1-r)-r}\left(a-a^r\right)\right)}\right)P(x)^r \to 0 \quad as \quad s\to\infty$$

for all $x\in E_1$. So we can conclude that $A(x)=A^{'}(x), \quad \forall x\in E_1$. This proves the uniqueness of A . Thus the mapping $A: E_1 \to E_2$ is a unique additive mapping satisfying (3.8).

Corollary 3.4

Let r be positive real numbers with $r<\frac{1}{2}$, and let $f: E_1 \to E_2$ be an odd mapping such that

$$\|D f(x,y)\|\le\begin{cases} P(x)^r P(y)^r , \\ P(x)^{2r}+P(y)^{2r}+P(x)^r P(y)^r , \end{cases}$$

for all $x,y\in E_1$. Then there exists a unique additive mappings $A: E_1 \to E_2$ satisfying

$$\|f(x)-A(x)\|\le\begin{cases} \lambda_2 P(x)^{2r} , \\ 3\lambda_2 P(x)^{2r} , \end{cases}$$

where $\lambda_2=\frac{1}{2^{2r+1}\left(a-a^{2r}\right)}$, for all $x\in E_1$.

IV. HYERS – ULAM STABILITY OF THE FUNCTIONAL EQUATION (1.5) AN EVEN MAPPING CASE

In this Section, we prove Hyers-Ulam Stability of the functional equation $Df(x,y)=0$ in paranormed spaces : an even mapping

Theorem 4.1

Let r and θ be positive real numbers with $r>2$, and let $f:E_2\to E_1$ be an even mapping satisfying (3.1). Then there exists a unique Quadratic mapping $Q:E_2\to E_1$ such that

$$P\left[f(x)-Q(x)\right]\le\frac{\theta}{2^r}\left[\frac{a^r}{a^r-a^2}\right]\|x\|^r \quad , \quad \forall x\in E_2$$

Proof . Using evenness of f in (3.1), we obtain

$$P\left[2a^2\left(f\left(\frac{x+y}{a}\right)+f\left(\frac{x-y}{a}\right)\right)-2\left(f(x+y)+f(x-y)\right)\right]\le\theta\left(\|x\|^r+\|y\|^r\right) \qquad (4.1)$$

for all $x,y\in E_2$. Replace (x,y) by (z,z) in (4.1), we obtain

$$P\left[2a^2 f\left(\frac{2z}{a}\right)-2f(2z)\right]\le\theta 2\|z\|^r,\quad \forall z\in E_2 \qquad (4.2)$$

Replace z by $\frac{x}{2}$ in (4.2) , we obtain

$$P\left[f(x)-a^2 f\left(\frac{x}{a}\right)\right]\le\frac{\theta}{2^r}\|x\|^r \quad , \qquad \forall x\in E_2$$

The rest of the proof is similar to the proof of theorem 3.1

Corollary 4.2

Let r and θ be positive real numbers with $r>1$, and let $f:E_2\to E_1$ be an even mapping such that

$$P\left[D f(x,y)\right]\le\begin{cases}\theta\left(\|x\|^r\|y\|^r\right),\\ \theta\left(\|x\|^{2r}+\|y\|^{2r}+\|x\|^r\|y\|^r\right),\end{cases}$$

for all $x,y\in E_2$. Then there exists a unique quadratic mappings $Q:E_2\to E_1$ satisfying

$$P\left[f(x)-A(x)\right]\le\begin{cases}\lambda_3\|x\|^{2r},\\ 3\lambda_3\|x\|^{2r},\end{cases}$$

where $\lambda_3=\dfrac{\theta a^{2r}}{2^{2r+1}\left(a^{2r}-a^2\right)}$, for all $x\in E_2$.

Theorem 4.3

Let r be positive real numbers with $r<2$, and let $f:E_1\to E_2$ be an even mapping satisfying (3.7) Then there exists a unique quadratic mapping $Q:E_1\to E_2$ such that

$$\left\| f(x) - Q(x) \right\| \le \frac{1}{a^2 - a^r}\left(\frac{a}{2}\right)^r P(x)^r, \qquad \forall x \in E_1 \quad .$$

Proof . Using evenness of f in (3.7), we obtain

$$\left\| 2a^2\left(f\left(\frac{x+y}{a}\right) + f\left(\frac{x-y}{a}\right)\right) - 2\left(f(x+y) + f(x-y)\right)\right\|$$

$$\le P(x)^r + P(y)^r, \quad \forall x, y \in E_1 \qquad (4.3)$$

Replace (x, y) by (z, z) in (4.3), we obtain

$$\left\| 2a^2 f\left(\frac{2z}{a}\right) - 2f(2z)\right\| \le 2P(z)^r \quad \forall z \in E_1 \qquad (4.4)$$

Replace z by $\frac{y}{2}$ in (4.4), we obtain

$$\left\| a^2 f\left(\frac{y}{a}\right) - f(y)\right\| \le \frac{1}{2^r} P(y)^r \qquad \forall y \in E_1 \qquad (4.5)$$

Again, replacing y by ax in (4.5), we obtain

$$\left\| f(x) - \frac{1}{a^2} f(ax)\right\| \le \frac{1}{a^{2-r}\, 2^r} P(x)^r \qquad \forall x \in E_1$$

The rest of the proof is similar to the proof of theorem 3.2

Corollary 4.4

Let r be positive real numbers with $r < 1$, and let $f : E_1 \to E_2$ be an even mapping such that

$$\left\| D f(x, y)\right\| \le \begin{Bmatrix} P(x)^r\, P(y)^r, \\ P(x)^{2r} + P(y)^{2r} + P(x)^r\, P(y)^r, \end{Bmatrix}$$

for all $x, y \in E_1$. Then there exists a unique quadratic mappings $Q : E_1 \to E_2$ satisfying

$$\left\| f(x) - Q(x)\right\| \le \begin{Bmatrix} \lambda_4\, P(x)^{2r}, \\ 3\lambda_4 P(x)^{2r}, \end{Bmatrix}$$

where $\lambda_4 = \dfrac{a^{2r}}{2^{2r+1}\left(a^2 - a^{2r}\right)}$, for all $x \in E_1$.

Theorem 4.5

Let r be positive real numbers with $r > 2$, and let $f : E_2 \to E_1$ be a mapping satisfying (3.1). Then there exists a unique additive mapping $A : E_2 \to E_1$ and quadratic mapping $Q : E_2 \to E_1$ such that

$$P\left[f(x) - A(x) - Q(x)\right] \le \theta\left(\frac{a}{2}\right)^r \left[\frac{1}{a\left(a^r - 1\right)} + \frac{1}{a^r - a^2}\right]\left\| x\right\|^r \qquad \forall x \in E_2 .$$

Theorem 4.6

Let r be positive real numbers with $r<1$, and let $f:E_1 \to E_2$ be a mapping satisfying (3.7) Then there exists a unique additive mapping $A:E_1 \to E_2$ and quadratic mapping $Q:E_1 \to E_2$ such that

$$\left\| f(x)-A(x)-Q(x)\right\| \le \frac{2}{a-a^r}\left(\frac{a}{2}\right)^r P(x)^r, \qquad \forall x\in E_1 \quad .$$

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