Mathematical Model for the dynamic behavior of two competing plant species

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Abstract- This work focuses on the mathematical model for the dynamic behavior of two competing plant species. We introduced a control mechanism and a growth enhancer to reduce the population of the weed and enhance the growth of the foodcrop respectively. We obtained steady state solutions which were characterized using the Linearization method. The results obtained were analyzed.

Index Terms- Mathematical model, competition, control mechanism, growth enhancer

I. Introduction

This article guides a stepwise walkthrough by Experts for writing a successful journal or a research paper starting from inception of ideas till their publications. Research papers are highly recognized in scholar fraternity and form a core part of PhD curriculum. Research scholars publish their research work in leading journals to complete their grades. In addition, the published research work also provides a big weight-age to get admissions in reputed varsity. Now, here we enlist the proven steps to publish the research paper in a journal.

Over the years, it has been difficult to predict the consequences of population growth of agricultural plants and this posed a problem to ecologist. Mathematical modeling has been a tool used by ecologist and has provided a reasonable approach to ecologist in the study of the interaction of species.

Modeling the population of species was first built by Malthus Thomas, Verhulst, Lotka and Volterra. In 1798, Malthus proposed the exponential growth model which stated that the population of species will grow unboundedly but this notion is unrealistic because the growth of one plant will affect the growth of the other. Due to the unrealistic notion of Malthus model, there was a need for modification and so in the year 1838, Verhulst derived the logistic model by introducing the notion of a carrying capacity and the interaction of plant and itself.

In the twentieth century, Lotka and volterra derived the Lotka-Volterra model for species competition. The model included the competition between two or more species called inter-specific competition. This model is of great interest to ecologist because of its biologically meaningful parameters.

It is known that in weed-crop competition, It is important to focused on reducing the effect of weed on food crop and to develop weed management control system. (kproff, 1993). Andrew et al, (2007) worked on the influence of canopy partitioning on the outcome of competition between plant species.

Yubin et al,(2011) worked on how to stabilize a mathematical model of the interactions of n-species population. Error estimation was also obtained Mark et al (2014) considered the mathematical model for biological invasion

Freekelston et al (2000) studied the determinants of the abundance of invasive annual weed. The dynamic of an annual pasture community were described from a five-year experimental and monitory study. Christian (1998) studied the analysis of plant competition experiments with variable plant density based on three published data sets. Hypotheses on the shape of the response surface curves, the number of necessary parameters and the effects of competition on plant growth were also tested using the maximum likelihood ration test. Mead (1967) considered the estimation of interplant competition. He was able to produce a new method of estimating the correlation coefficient.

Neville et al, (2007) considered the complex interaction between two plant species in a harsh environment. They considered how climate change affects the growth of plants. The results were characterized, analyzed and plotted .

II. Mathematical formulation of model

We therefore consider two species Lotka Volterra Competition model with species having logistic growth in the Lotka Volterra system which makes it more realistic. The interaction of the two plant species competing for same limited resources leads to both plant species experiencing intra-specific and inter-specific competition and this is described by the following ordinary differential equations;

$$\frac{dS}{dt} = r_1 S\left(1 - \frac{S}{k_1} - \frac{b_{sp}P}{k_1} + \gamma\right) \qquad (1)$$

$$\frac{dP}{dt} = r_2 P\left(1 - \frac{P}{k_2} - \frac{b_{ps}S}{k_2} - \delta\right) \qquad (2)$$

Where S(t) is density of foodcrop at time (t) which is the desired plant p(t) is density of weed (undesirable plant) at time (t).

r, and r_2 are intrinsic growth rate.K_1 and k_2 are the carrying capacity of the foodcrop and weed respectively b_{sp} measures the competitive effect of P on S. b_{ps} measures the competitive effect of S on P.

Note that: $r_1 S\left(1-\frac{S}{k_1}-\frac{b_{sp}P}{k_1}\right)$ and $r_2 P\left(1-\frac{P}{k_2}-\frac{b_{ps}S}{k_2}\right)$ obey the logistic equation where each species inhibits its own growth through intra-specific competition and inter specific between the two plant species. δ represent the control mechanism γ and growth enhancer inter-specific competition between the two plant species.Dues to the competition between the plant species for limited same resources which will lead to the extinction of one the plant species, a control mechanism (herbicide) is applied to the weed by introducing the parameter δ of the weed. Since the herbicides could affect the nutrients we thus propose to introduce a growth enhancer (fertilizer) which is appplied to the foodcrop and the growth enhancer parameter γ will enhance the growth of the foodcrop and thus increase the population of the foodcrop.

(1) Steady State Solution

A Steady state solution exist if

$$\frac{dS}{dt}=F_1(S,P)=0 \qquad (3)$$

$$\frac{dP}{dt}=F_2(S,P)=0 \qquad (4)$$

We introduce non-dimensional quantities by writing.

$$U_1=\frac{S}{K_1},\quad U_2=\frac{P}{K_2},\quad \tau=r_1 t,$$

$$\rho=\frac{r_2}{r_1},\quad a_{SP}=b_{SP}\frac{K_2}{K_1},\quad a_{PS}=b_{PS}\frac{K_1}{K_2},$$

Using the non-dimensional quantities

$$\tau=r_1 t$$

We obtain,

$$\frac{dU_1}{d\tau}=\rho U_1\left(1-U_1-a_{SP}U_2+\gamma\right) \qquad (5)$$

$$\frac{dU_2}{d\tau}=\rho U_2\left(1-U_2-a_{PS}U_1-\delta\right) \qquad (6)$$

We now proceed to obtain the steady state solutions of the ordinary differential equations.

Let U^*_1 and U^*_2 be arbitrary steady-state solutions such that

$$f_1\left(U^*_1,U^*_2\right)=f_2\left(U^*_1,U^*_2\right)=0 \qquad (7)$$

$$f_1\left(U^*_1,U^*_2\right)=U^*_1\left(1-U^*_1-a_{sp}U^*_2+\gamma\right)=0 \qquad (8)$$

$$f_2\left(U^*_1,U^*_2\right)=\rho U^*_2\left(1-U^*_2-a_{ps}U^*_1-\delta\right)=0$$

Using the above, we obtain the following steady state solutions:

Case 1: The trivial case when $U^*_1 = 0$ and $U^*_2 = 0$.

∴ (0, 0) is trivial S.S.S

Case 2:

Assume $U_{*1} \neq 0$ and $U^*_2 = 0$

∴ $(1+\gamma, 0)$ is S.S.S

Case 3:

If $U^*_1 = 0$ and $U^*_2 \neq 0$

(0,1) is a S.S.S

Case 4:

If $\left(U^*_1=1 and U^*_2\neq 0\right)$ then

$\left(1,\frac{\gamma}{a_{sp}}\right)$ is a S.S.S.

Case 5:

If $\left(U^*_1\neq 0 and U^*_2\neq 0\right)$

$$\left(\frac{1+\gamma-(1-\delta)a_{sp}}{1-a_{sp}a_{ps}},\frac{1-(1+\gamma)a_{ps}-\delta}{1-a_{sp}a_{ps}}\right)$$

(2) Characterization of Steady – state Solution

We Linearise the interaction functions of equations (1) and (2)) by obtaining the Jacobian Matrix in order to investigate the stabilities of the steady state solution. The characteristics equation is given by

det $(J-\lambda I)=0$ where,

$$J=\begin{pmatrix}\frac{\partial f_1}{\partial U^*_1} & \frac{\partial f_1}{\partial U^*_2}\\ \frac{\partial f_2}{\partial U^*_1} & \frac{\partial f_2}{\partial U^*_2}\end{pmatrix} \qquad (9)$$

$$f_1\left(U^*_1,U^*_2\right)=U^*_1\left(1-U^*_1-a_{sp}U^*_2+\gamma\right)=0 \qquad (10)$$

$$f_2\left(U^*_1,U^*_2\right)=\rho U^*_2\left(1-U^*_2-a_{ps}U^*_1-\delta\right)=0$$

$$\frac{\partial f_1}{\partial U_1^*} = 1 - 2U_1^* - a_{sp}U_2^* + \gamma$$
$$\frac{\partial f_1}{\partial U_2^*} = -a_{sp}U_1^* \qquad (11)$$
$$\frac{\partial f_2}{\partial U_1^*} = -\rho a_{ps}U_2^*$$
$$\frac{\partial f_2}{\partial U_2^*} = \rho - 2\rho U_2^* - \rho a_{ps}U_1^* - \rho\delta$$

For the trivial S.S.S (0.0)

$$J = \begin{pmatrix} 1+\gamma & 0 \\ 0 & \rho(1-\delta) \end{pmatrix}$$

$$J - \lambda = \begin{pmatrix} 1+\gamma-\lambda & 0 \\ 0 & \rho(1-\delta)-\lambda \end{pmatrix}$$
$$\lambda_1 = 1+\rho$$
$$\lambda_2 = \rho(1-\delta)$$

The trivial S.S.S is unstable if $\rho \succ \rho\delta$.

For S.S.S (1 + γ, 0)

$$J = \begin{pmatrix} 1-2+2\gamma & -a_{sp}-a_{sp}\gamma \\ 0 & \rho-\rho a_{ps}-\rho a_{ps}\gamma-\rho\delta \end{pmatrix}$$

$$J-\lambda = \begin{pmatrix} 1-2+3\gamma-\lambda & -a_{sp}-a_{sp}\gamma-\lambda \\ 0 & \rho-\rho a_{ps}-\rho a_{ps}\gamma-\rho\delta-\lambda \end{pmatrix}$$
$$\det(J-\lambda I) = \begin{pmatrix} 1-2+3\gamma-\lambda & -a_{sp}-a_{sp}\gamma \\ 0 & \rho-\rho a_{ps}-\rho a_{ps}\gamma-\rho\delta-\lambda \end{pmatrix}$$

$$\det(J-\lambda I) = \begin{pmatrix} -1+3\gamma-\lambda & -a_{sp}(1+\gamma) \\ 0 & \rho-\rho a_{ps}-\rho a_{ps}\gamma-\rho\delta-\lambda \end{pmatrix} \qquad (12)$$

Using the upper triangle matrix, we obtain the eigenvalues
$$\lambda_1 = 3\gamma - 1$$
$$\lambda_2 = \rho - \rho a_{ps}(\gamma+1) - \rho\delta$$

For S.S.S $\left(1, \frac{\gamma}{a_{sp}}\right)$

$$J = \begin{pmatrix} -1 & -a_{sp} \\ \frac{-\gamma\rho a_{ps}}{a_{sp}} & \rho(1-\frac{2\gamma}{a_{sp}}-a_{ps}-\delta) \end{pmatrix} \qquad (13)$$

Using the eigenvalue formula for 2x 2 system
$$\lambda^2 - tr(A)\lambda + \det(A)I \qquad (14)$$
$$tr(A) = -1 + \rho(1-\frac{2\gamma}{a_{sp}} - a_{ps} - \delta)$$

$$\det(A) = -\rho + \frac{2\gamma\rho}{a_{sp}} + \rho a_{ps}(1-\gamma) + \rho\delta$$

Thus,
$$\lambda^2 - (-1+\rho(1-\frac{2\gamma}{a_{sp}} - a_{ps} - \delta))\lambda + \frac{2\gamma\rho}{a_{sp}} +$$
$$+ \rho a_{ps}(1-\gamma) + \rho\delta - \rho$$
$$\lambda^2 + (1-\rho+\frac{2\rho\gamma}{a_{sp}} + \rho a_{ps} + \rho\delta)\lambda + \frac{2\gamma\rho}{a_{sp}} +$$
$$+ \rho a_{ps}(1-\gamma) + \rho\delta - \rho \qquad (15)$$

Using factorization method, we obtain

$$\lambda_{12} = -\frac{1}{2}\{(1-\rho+\frac{2\rho\gamma}{a_{sp}} + \rho a_{ps} + \rho\delta) \pm$$
$$\sqrt{(1-\rho+\frac{2\rho\gamma}{a_{sp}} + \rho a_{ps} + \rho\delta)^2 - 4(\frac{2\gamma\rho}{a_{sp}} + \rho a_{ps}(1-\gamma) + \rho\delta - \rho}\}$$

The steady state is stable if and only if $\lambda_1 < 0$ and $\lambda_2 < 0$ but this is only possible if
$$(1-\rho+\frac{2\rho\gamma}{a_{sp}} + \rho a_{ps} + \rho\delta)^2 - 4(\frac{2\gamma\rho}{a_{sp}} +$$
$$+ \rho a_{ps}(1-\gamma) + \rho\delta - \rho \quad \mathbf{> 0}$$

ForS.S.S $\left(\frac{1+\gamma-(1-\delta)a_{sp}}{1-a_{sp}a_{ps}}, \frac{1-(1+\gamma)a_{ps}-\delta}{1-a_{sp}a_{ps}}\right)$

The Jacobian matrix is

$$J = \begin{pmatrix} \dfrac{-1+3\delta a_{sp} + +2\gamma a_{sp}a_{ps} + a_{sp} - \gamma}{1-a_{sp}a_{ps}} & \dfrac{-a_{sp} - \gamma a_{sp} + a_{sp}^{2} - -\delta a_{sp}^{2}}{1-a_{sp}a_{ps}} \\ \dfrac{-\rho a_{ps} + \rho a_{ps}^{2} + \gamma\rho a_{ps}^{2} + +\rho\delta a_{ps}}{1-a_{sp}a_{ps}} & \dfrac{-\rho + \rho a_{ps} + 2\gamma\rho a_{ps} + +\rho\delta}{1-a_{sp}a_{ps}} \end{pmatrix}$$

Similarly, using the eigenvalue system formula we obtain

$$\lambda_{12} = -\frac{1}{2(1-a_{sp}a_{ps})}\{(-1+3\delta a_{sp} + +2\gamma a_{sp}a_{ps} + a_{sp} - \gamma - \rho + \rho a_{ps} + 2\gamma\rho a_{ps} + +\rho\delta) \pm \sqrt{\begin{array}{l}(-1+3\delta a_{sp} + \\ +2\gamma a_{sp}a_{ps} + a_{sp} - \gamma - \rho + \rho a_{ps} + 2\gamma\rho a_{ps} + \\ +\rho\delta)^2 - 4(-a_{sp} - \gamma a_{sp} + a_{sp}^{2} - \\ -\delta a_{sp}^{2})(-\rho a_{ps} + \rho a_{ps}^{2} + \gamma\rho a_{ps}^{2} + \\ +\rho\delta a_{ps})\end{array}}\}$$

The steady state is stable if and only if $(-1+3\delta a_{sp} + 2\gamma a_{sp}a_{ps} + a_{sp} - \gamma - \rho + \rho a_{ps} + 2\gamma\rho a_{ps} + +\rho\delta)^2 - 4(-a_{sp} - \gamma a_{sp} + a_{sp}^{2} - \delta a_{sp}^{2})(-\rho a_{ps} + \rho a_{ps}^{2} + \gamma\rho a_{ps}^{2} + +\rho\delta a_{ps}) > \mathbf{0}$ and $a_{sp}, a_{ps} < 1$.

Note that, the stability of the steady state depends on the size of $\rho, \gamma, \delta, a_{sp,} a_{ps}$.

III. Results

From above, we obtained four stable solutions which are dependent on the size $\rho, \gamma, \delta, a_{sp,} a_{ps}$. By setting:

1. $a_{sp} > 1, a_{ps} < 1, \quad \delta > \rho, \gamma > \rho$ the foodcrop becomes dominant while the weed goes into extinction.
2. $a_{sp} < 1, a_{ps} > 1,$ the foodcrop goes into extinction while the weed becomes dominant.
3. $a_{sp} < 1, a_{ps} < 1,$ the two plants coexist.
4. $a_{sp} > 1, a_{ps} > 1,$ one of the plants will go into extinction while the other remains dominant.

To prevent the weed from out- compete the foodcrop in (3) and (4), we set $\delta > \rho, \gamma > \rho$. That is the control mechanism will reduce the population of the weed and the growth enhancer improves the growth rate of the foodcop.

IV. Conclusion

- Indeed Mathematical model has provided a good and reasonable approach to ecologist in the study of the interaction of plant species. Having considered the Lotka Volterra competition model with the introduction a growth enhancer and a control mechanism, we obtained five steady state solutions which were characterized using the Linearization method. We can say that the size of the parameters $\rho, \gamma, \delta, a_{sp,} a_{ps}$ will determine the stability of the steady state solutions. Our interest is to ensure that the weed does not out-compete the foodcrop so we set $\delta > \rho, \gamma > \rho$.

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