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Split-Deflection Method of Classical Rectangular Plate Analysis

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Abstract- This paper presents an split-deflection method of classical rectangular plate analysis. In this method, the deflection was split into x and y components of deflection. It was assumed that the deflection of the rectangular plate is the product of these two components. With this assumption, total potential energy functional was derived from principles of theory of elasticity. By direct variation of the potential energy functional, direct governing equation was obtained. Polynomial deflection and trigonometric deflection were respectively used for the x and y components of the deflection for ssss plate. This deflection components were substituted into this direct governing equation and the coefficient of the deflection was obtained to be 0.0132629qa4/D. With this coefficient, the maximum central deflection of the plate was obtained to be 0.0041446qa4/D. This was compared with the maximum central deflection of Navier's (0.00416qa4/D) and Levi's(0.00406qa4/D) plates. It was observed that the value from the present study makes lower bound and upper bound differences of 0.37% and 2.08% with the values from Navier and Levi respectively. This shows that this present method is reliable.

Index Terms- Direct variation, split-deflection, total potential energy functional, polynomial function, trigonometric function, deflection

I. Introduction

The common energy methods in classical plate theory analysis include Galerkin's, Raleigh, Raleigh-Ritz, Ritz methods etc. (Njoku et al., 2013, Ibearugbulem et al., 2014, Ibeabuchi, 2014). These methods use orthogonal functions in form double Furrier series (that product of two mutually perpendicular trigonometric functions) or orthogonal polynomial. The energy equations of the methods used earlier than now are based on the orthogonal deflection functions (Hutchinson, 1992, Jianqiao, 1994, Ugural, 1999, Ventsel and Krauthammer, 2001, Wang et al., 2002, Taylor and Govindjee, 2004, Szilard, 2004, Jiu et al., 2007, Erdem et al., 2007, Ezeh et al., 2013, Ibearugbulem, 2014). However, none of the previous energy methods have used spited deflection functions - that is a deflections function that is typically separated into two independent distinct functions ($w = w_x * w_y$). In this case w_x may be polynomial and w_y may be trigonometry. In similar way stress and strain terms are also based on individual components of w (deflection) for example using:

$$\varepsilon_x = \frac{du}{dx} = -z\frac{d^2w_x}{dx^2}w_y \text{ instead of } \varepsilon_x = -z\frac{d^2w}{dx^2}$$

$$\gamma_{xy} = -2z\frac{dw_x}{dx}\frac{dw_y}{dy} \text{ instead of } \gamma_{xy} = -2z\frac{d^2w}{dxdy}$$

The main reason for this modification is to help the analysis who may have difficulty in obtaining orthogonal function for a plate of a particular boundary condition. In this case, the analyst who may have easy access to deflection equations for beams of any boundary condition can find the proposed method quite useful and handy.

SPLIT-DEFLECTION

The assumption here is that the general deflection, w is split into w_x and w_y. That is:

$$w = w_x . w_y \qquad 1$$

Where the x and y components of the deflection are defined as:

$$w_x = \sqrt{A}.h_1 \qquad 2$$

$$w_y = \sqrt{A}.h_2 \qquad 3$$

Substituting equations (2) and (3) into equation (1) gives:

$$w = A\,h_1 h_2 \qquad 4$$

IN-PLANE DISPLACEMENTS

From the assumption that vertical shear strains are zero for classical plate, we obtain:

$$u = -z\frac{dw}{dx} = -z\frac{dw_x}{dx}w_y \qquad 5$$

$$v = -z\frac{dw}{dy} = -z\frac{dw_y}{dy}w_x \qquad 6$$

STRAIN DEFLECTION RELATIONSHIP

The three in-plane strains of classical plate are obtained by using equations (5) and (6):

$$\varepsilon_x = \frac{du}{dx} = -z\frac{d^2w_x}{dx^2}w_y \qquad 7$$

$$\varepsilon_y = \frac{dv}{dy} = -z\frac{d^2w_y}{dy^2}w_x \qquad 8$$

$$\gamma_{xy} = \frac{du}{dy} + \frac{dv}{dx} = -2z\frac{dw_x}{dx}\frac{dw_y}{dy} \qquad 9$$

STRESS – STRAIN RELATIONSHIP

The in-plane constitutive equations are:

$$\sigma_x = \frac{E}{1-\mu^2}\left[\varepsilon_x + \mu\varepsilon_y\right] \qquad 10$$

$$\sigma_y = \frac{E}{1-\mu^2}\left[\mu\varepsilon_x + \varepsilon_y\right] \qquad 11$$

$$\tau_{xy} = \frac{E(1-\mu)}{2(1-\mu^2)}\gamma_{xy} \qquad 12$$

STRESS – DEFLECTION RELATIONSHIP

Substituting equations (7), (8) and (9) into equations (10), (11) and (12) where appropriate gives:

$$\sigma_x = \frac{-Ez}{1-\mu^2}\left[\frac{d^2w_x}{dx^2}w_y + \mu\frac{d^2w_y}{dy^2}w_x\right] \qquad 13$$

$$\sigma_y = \frac{-Ez}{1-\mu^2}\left[\mu\frac{d^2w_x}{dx^2}w_y + \frac{d^2w_y}{dy^2}w_x\right] \qquad 14$$

$$\tau_{xy} = \frac{-Ez(1-\mu)}{(1-\mu^2)}\frac{dw_x}{dx}\frac{dw_y}{dy} \qquad 15$$

TOTAL POTENTIAL ENERGY

The strain energy is defined as:

$$U = \frac{1}{2}\int_x\int_y\left[\int_{-\frac{t}{2}}^{\frac{t}{2}}[\sigma_x\varepsilon_x + \sigma_x\varepsilon_x + \sigma_x\varepsilon_x]dz\right]dx\,dy \quad 16$$

For pure bending analysis , the external work is given as:

$$V = \int_x\int_y q\,w_x.w_y\,dx\,dy$$

That is

$$V = q\int_x w_x dx\int_y w_y\,dy \qquad 17$$

Substituting equations (10) to (15) into equation (16) gives strain energy – deflection relationship as:

$$U = \frac{D}{2}\int_x\int_y\left[\left(\frac{d^2w_x}{dx^2}\right)^2 w_y^2 + 2\left(\frac{dw_x}{dx}\right)^2\left(\frac{dw_y}{dy}\right)^2 + \left(\frac{d^2w_y}{dy^2}\right)^2 w_x^2\right]dx\,dy$$

That is

$$U = \frac{D}{2}\left[\int_x\left(\frac{d^2w_x}{dx^2}\right)^2dx\int_y w_y^2\,dy\right] + \frac{2D}{2}\left[\int_x\left(\frac{dw_x}{dx}\right)^2dx\int_y\left(\frac{dw_y}{dy}\right)^2dy\right] + \frac{D}{2}\left[\int_x w_x^2\,dx\int_y\left(\frac{d^2w_y}{dy^2}\right)^2dy\right] \qquad 18$$

Subtracting equation (17) from Equation (18) gives the total potential energy functional as:

$$\Pi = \frac{D}{2}\left[\int_x\left(\frac{d^2w_x}{dx^2}\right)^2dx\int_y w_y^2\,dy\right] + \frac{2D}{2}\left[\int_x\left(\frac{dw_x}{dx}\right)^2dx\int_y\left(\frac{dw_y}{dy}\right)^2dy\right] + \frac{D}{2}\left[\int_x w_x^2\,dx\int_y\left(\frac{d^2w_y}{dy^2}\right)^2dy\right] - q\int_x w_x dx\int_y w_y\,dy \qquad 19$$

Substituting equations (1) and (2) into equation (19) gives:

$$\Pi = \frac{A^2D}{2}\left[\int_x\left(\frac{d^2h_1}{dx^2}\right)^2dx\int_y h_2^2\,dy\right] + \frac{2A^2D}{2}\left[\int_x\left(\frac{dh_1}{dx}\right)^2dx\int_y\left(\frac{dh_2}{dy}\right)^2dy\right] + \frac{A^2D}{2}\left[\int_x h_1^2\,dx\int_y\left(\frac{d^2h_2}{dy^2}\right)^2dy\right] - qA\int_x h_1dx\int_y h_2\,dy \qquad 20$$

Now, equation (20) can be written in non dimensional axes R and Q.

$$x = aR \qquad 21$$

$$y = aQ \qquad 22$$

$$P = b/a \qquad 23$$

Where a, b and P are the plate lengths in x and y axes and long span- short span aspect ratio respectively.

Substituting equations (21), (22) and (23) into equation (20) gives:

$$\Pi = \frac{abA^2D}{2a^4}\left[\int_0^1\left(\frac{d^2h_1}{dR^2}\right)^2dR\int_0^1 h_2^2\,dQ\right] + 2\frac{abA^2D}{2a^4P^2}\left[\int_0^1\left(\frac{dh_1}{dR}\right)^2dR\int_0^1\left(\frac{dh_2}{dQ}\right)^2dQ\right] + \frac{abA^2D}{2a^4P^4}\left[\int_0^1 h_1^2\,dR\int_0^1\left(\frac{d^2h_2}{dQ^2}\right.\right.$$

DIRECT VARIATION OF TOTAL POTENTIAL ENERGY

Equation (24) shall be differentiated with respect to the deflection coefficient, A gives:

$$\frac{d\Pi}{dA} = \frac{AD}{a^4}\left[\int_0^1\left(\frac{d^2h_1}{dR^2}\right)^2dR\int_0^1 h_2^2\,dQ\right] + 2\frac{AD}{a^4P^2}\left[\int_0^1\left(\frac{dh_1}{dR}\right)^2dR\int_0^1\left(\frac{dh_2}{dQ}\right)^2dQ\right] + \frac{AD}{a^4P^4}\left[\int_0^1 h_1^2\,dR\int_0^1\left(\frac{d^2h_2}{dQ^2}\right)^2dQ\right] - q\int_0^1 h_1dR\int_0^1 h_2\,dQ = 0$$

That is

$$\frac{AD}{a^4}\left[\int_0^1\left(\frac{d^2h_1}{dR^2}\right)^2dR\int_0^1 h_2^2\,dQ\right] + 2\frac{AD}{a^4P^2}\left[\int_0^1\left(\frac{dh_1}{dR}\right)^2dR\int_0^1\left(\frac{dh_2}{dQ}\right)^2dQ\right] + \frac{AD}{a^4P^4}\left[\int_0^1 h_1^2\,dR\int_0^1\left(\frac{d^2h_2}{dQ^2}\right)^2dQ\right] = q\int_0^1 h_1dR\int_0^1 h_2\,dQ \qquad 25$$

This equation (25) is the direct governing equation of rectangular plate under pure bending from this present method. Rearranging equation (25) and making the coefficient of deflection the subject of the formula gives:

$$A = \frac{\frac{qa^4}{D}\left(\int_0^1 h_1 dR \int_0^1 h_2 \; dQ\right)}{k_x + 2\frac{k_{xy}}{p^2} + \frac{k_y}{p^4}} \qquad 26$$

Where

$$k_x = \int_0^1 \left(\frac{d^2h_1}{dR^2}\right)^2 dR \int_0^1 h_2{}^2 \, dQ \qquad 27$$

$$k_{xy} = \int_0^1 \left(\frac{dh_1}{dR}\right)^2 dR \int_0^1 \left(\frac{dh_2}{dQ}\right)^2 dQ \qquad 28$$

$$k_y = \int_0^1 h_1{}^2 \, dR \int_0^1 \left(\frac{d^2h_2}{dQ^2}\right)^2 dQ \qquad 29$$

NUMERICAL EXAMPLE

Analyze a classical rectangular thin isotropic plate with all the four edge simply supported subjected to uniform distributed load, q. Let the x and y components of deflection be:

$$w_x = \sqrt{A}\,(R - 2R^3 + R^4) \qquad 30$$

$$w_y = \sqrt{A}\,\text{Sin}\,\pi Q \qquad 31$$

From equations (30) (31), h_1 and h_2 are:

$$h_1 = R - 2R^3 + R^4 \qquad 32$$

$$h_2 = \text{Sin}\,\pi Q \qquad 33$$

With these we obtain:

$$\int_0^1 h_1 dR = 0.02 \text{ and } \int_0^1 h_2 dQ = 0.127273\text{. Hence,}$$

$$\int_0^1 h_1 dR \int_0^1 h_2 \; dQ = (0.2)\left(\frac{2}{\pi}\right) = 0.127273 \qquad 34$$

$$\int_0^1 \left(\frac{d^2h_1}{dR^2}\right)^2 dR = 4.8 \text{ and } \int_0^1 h_2{}^2 \, dQ = 0.5\text{. Hence,}$$

$$k_x = (4.8)(0.5) = 2.4 \qquad 35$$

$$\int_0^1 \left(\frac{dh_1}{dR}\right)^2 dR = 0.4857142 \text{ and } \int_0^1 \left(\frac{dh_2}{dQ}\right)^2 dQ = 0.5\pi^2.$$

$$\text{Hence,} \quad k_{xy} = (0.4857142)(0.5\pi^2) = 2.399 \qquad 36$$

$$\int_0^1 h_1{}^2 \, dR = 0.0492063 \text{ and } \int_0^1 \left(\frac{d^2h_2}{dQ^2}\right)^2 dQ = 0.5\pi^4.$$

$$\text{Hence, } k_y = (0.0492063)(0.5\pi^4) = 2.4 \qquad 37$$

Substituting equations (34) to (29) into equation (37) gives

$$A = \frac{0.127273\,\frac{qa^4}{D}}{2.4 + \frac{4.798}{p^2} + \frac{2.4}{p^4}} \qquad 38$$

The maximum deflection shall occur at the center of the rectangular plate. At the center of the circle, R = Q = 0.5. Substituting o.5 into equations (32) and (33) gives:

$$h_1(0.5) = 0.3125 \qquad 39$$

$$h_2(0.5) = 1 \qquad 40$$

Substituting equations (38), (39) and (40) into equation (4) gives the equation for maximum deflection as:

$$w_{max} = \frac{0.0397728\,\frac{qa^4}{D}}{2.4 + \frac{4.798}{p^2} + \frac{2.4}{p^4}} \qquad 41$$

II. RESULTS AND CONCLUSIONS

The result of this present study is presented on table 1. A close critical observation of the table reveals that the maximum recorded percentage difference is 0.276. The result from past work was obtain from the work of Ibearugbulem et al. (2014). They used complete orthogonal polynomial deflection equation where as the present study used a part as polynomial function and the other as trigonometric function. This difference in the deflection equations between the previous study and the present one may be responsible for the maximum recorded percentage difference. Again, for aspect ratio of 1.0, the result from the present study is 0.00414qa4/D. On comparing this with values from works of Navier's (0.00416qa4/D) and Levi's(0.00406qa4/D), it was observed that the value from the present study makes lower bound and upper bound differences of 0.37% and 2.08% with the values from Navier and Levi respectively.

Thus, an inference that the results obtained herein is close to those of previous study is drawn. Hence, the present method as present herein is reliable and is recommended for use in classical plate analysis. For future studies, it is recommend that other aspects of thin plate analysis like buckling and free vibration should be carried out using the present method. It is also recommended that this present method is extended to refined plate theory analysis (RPT).

Table 1: Center deflection of ssss isotropic thin plate under uniform distributed load

Aspect ratio, P	Deflection at center (qa^4/D)		Percentage difference
	Present	Past (Ibearugbulem et al., 2014)	
1	0.00414	0.00414	0.000
1.1	0.00497	0.00496	0.202
1.2	0.00577	0.00576	0.174
1.3	0.00654	0.00653	0.153
1.4	0.00727	0.00725	0.276
1.5	0.00794	0.00793	0.126
1.6	0.00857	0.00856	0.117
1.7	0.00915	0.00913	0.219
1.8	0.00968	0.00966	0.207
1.9	0.01016	0.01015	0.099
2	0.01061	0.01059	0.189

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