Bifurcation Analysis and Fractal Dimensions of a Non-Linear Map

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Abstract- In this paper a two dimensional non-linear map is taken, whose various dynamic behavior is analyzed. Some useful numerical algorithms to obtain fixed points and bifurcation values of period 2^n, $n = 0,1,2...$ have been discussed. It has shown how the ratio of three successive period doubling bifurcation points ultimately converge to the Feigenbaum constant. This ascertains that the map follows the period doubling route to chaos. The parameter value where chaos starts is verified by lyapunov exponent. Further various fractal dimensions like Correlation dimension, Box-counting and Information dimension have been calculated to verify the geometry of the strange attractor.

Index Terms- Accumulation point , Feigenbaum Universal Constant , Fractal Dimension, Lyapunov exponent, Period-Doubling Bifurcation.

I. Introduction

Bifurcations, as a universal route to chaos, is one of the most exciting discoveries of the last few years in the field of nonlinear dynamical systems. The universality discovered by the elementary particle theorist, Mitchell J. Feigenbaum in 1975 in one-dimensional iterations with the logistic map, $x_{n+1} = r\, x_n(1 - x_n)$ has successfully led to discover that large classes of non-linear systems exhibit transitions to chaos which are universal and quantitatively measurable [6]. One of his fascinating discoveries is that if a family presents period doubling bifurcations then there is an infinite sequence $\{\mu_n\}$ of bifurcation values such that $\lim_{n\to\infty} \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n} = \delta$, where δ is a universal number which is now termed as Feigenbaum constant. [18]

Logistic map plays a role model in the study of dynamical system. It represents the class of low dimensional models of discrete time evolution which are easy to treat yet full of richness with dynamical behavior. M.Feigenbaum [6] discussed the universal behaviors of one dimensional unimodal maps of the form $x_{n+1}=\lambda f(x_n)$. Borges E.P. and Pessoa R W S have taken a generalized version of logistic map [15] replacing multiplication by the binary operation "q-product". Gottlieb [9] has discussed different dynamical properties in a generalization of logistic map using fractional exponents. Besides these generalized form of logistic map has got its application in economics, statistics, cryptography, biology e.t.c. [20, 21]

In this paper, we have taken two dimensional map of the form $f(x, y) = \left(ax^k(b - cx^r) + xy, x\right)$. The map is closely related with the generalized logistic map, however the generalized logistic model has been affected by the presence of another population represented by y. From mathematical point of view it is assumed that at some stage the newly added population represented by y maintain the same value as the population x recorded one stage earlier.

This paper is organized in the following manner. Section 1 presents the fixed points/ periodic points and bifurcation values of period $2^n, n = 0,1,2...$ with suitable numerical methods and how the ratio of three successive period doubling bifurcation points ultimately converge to the Feigenbaum constant. Also the bifurcation points give information about the value of the control parameter "a", where the onset of chaos occurs. In section 2 the accumulation point(onset of chaos) is calculated. In section 3 the calculated results are verified with the help of Lyapunov Exponent of the map. In section 4 generalized correlation sum has been considered, which helps in finding box counting dimensions, correlation dimensions, information dimensions to quantify the dimension of the attractors near the accumulation point.

II. BIFURCATION SCENARIO OF THE MAP

One of the main aim of bifurcation theory is to find out the fixed points, periodic points of maps and look for the region of their stability. We now fix some of the parameters say c, b, k, r and keep varying 'a' to analyse the detailed dynamical behaviour of the map. Let us take b=1, c=1, k=0.5, r=0.1 .On inspection it can be seen that (0,0) is a fixed point of the model satisfying the equation $f(x, y) = (x, y) = (ax^k(b - cx^r) + xy, x)$ ...... (1)

Using "Mathematica" we generate the bifurcation diagram for the observation of the whole dynamical behaviour of the map varying "a".

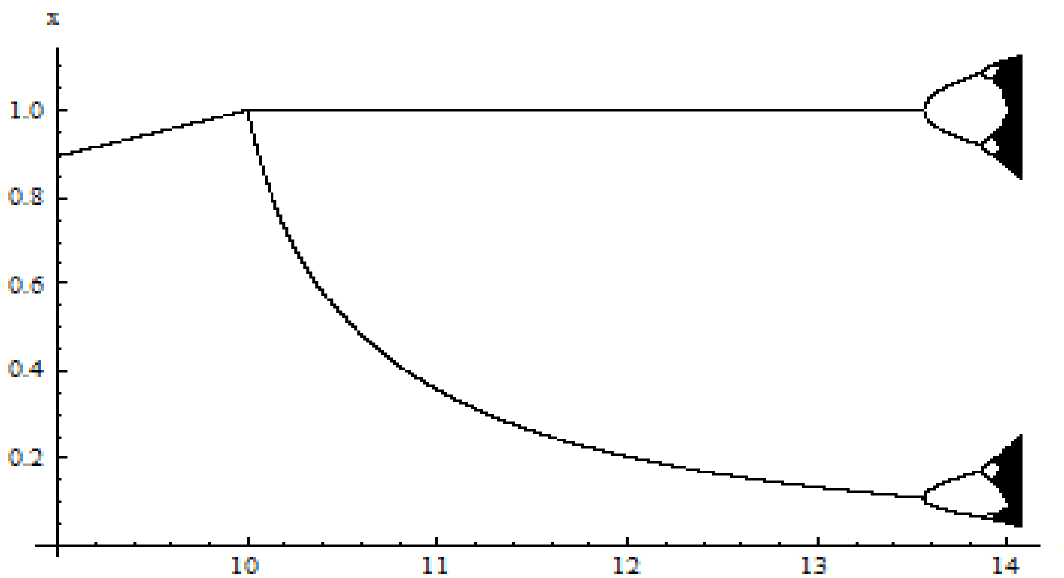

Figure 1: The figure is generated using 100 points which are taken after iterating 5000 points of the map at every parameter value of 'a', and plotted the x coordinate of the point (x,y) vs'a'.

It is a visual summery of the succession of period doubling produced as the parameter increases. Initially the map has one stable fixed point up to certain value of the parameter "a". The bifurcation diagram nicely shows the forking of the periods of stable orbits from 1 to 2, then 2 to 4 etc. The interesting thing about the diagram is that as the periods go to infinity, still the parameter remains finite. For further investigation numerical procedure is adopted to get the bifurcation point, which may help to confirm chaos. From the diagram it has been clear that the map follows period doubling route to chaos as "a" is varied.

A. Numerical Method for Obtaining Periodic Points

There are so many highly developed numerical algorithms to find a periodic fixed point. But the Newton Recurrence formula is one of the best numerical methods with negligible error for our purpose. Moreover, it gives fast convergence of a periodic fixed point.

The Newton Recurrence formula is

$\overline{x_{n+1}} = \overline{x_n} - Df(\overline{x_n})^{-1} f(\overline{x_n})$, where n = 0, 1, 2,... and $Df(\bar{x})$ is the Jacobian of the map f at the vector $\bar{x} = (x_1, x_2)$(say) . This map $f(x,y)$ of equation (1) is equal to $f^k - I$, where k is the appropriate period. The Newton formula actually gives the zero(s) of a map, and to apply this numerical tool in the map one needs a number of recurrence formulae which are given below [4].

Let the initial point be (x_0, y_0) and let $S(x,y) = ax^k(b - cx^r) + xy, T(x,y) = x$;

Proceeding in this manner the following recurrence formula

Let

$$A_0 = \left.\frac{\partial S}{\partial x}\right|_{(x_0,y_0)}, \quad B_0 = \left.\frac{\partial S}{\partial y}\right|_{(x_0,y_0)}, \quad C_0 = \left.\frac{\partial T}{\partial x}\right|_{(x_0,y_0)},$$

$$D_0 = \left.\frac{\partial T}{\partial y}\right|_{(x_0,y_0)}$$

And

$$A_k = \begin{pmatrix} \left.\frac{\partial S}{\partial x}\right|_{(x_k,y_k)} & \left.\frac{\partial S}{\partial x}\right|_{(x_k,y_k)} \\ \left.\frac{\partial T}{\partial x}\right|_{(x_k,y_k)} & \left.\frac{\partial T}{\partial y}\right|_{(x_k,y_k)} \end{pmatrix} \begin{pmatrix} A_{k-1} & B_{k-1} \\ C_{k-1} & D_{k-1} \end{pmatrix}$$

for all $k \geq 1$

Since the fixed point of the map f is a zero of the map

F(x,y) = f(x,y)-(x,y), the Jacobian of $F^{(k)}$ is given by

$$J_k - I = \begin{pmatrix} A_k - 1 & B_k \\ C_k & D_k - 1 \end{pmatrix}.$$

Its inverse is

$$(J_k - I)^{-1} = \frac{1}{\Delta}\begin{pmatrix} D_k - 1 & -B_k \\ -C_k & A_k - 1 \end{pmatrix}$$

Where $\Delta = (A_k - 1)(D_k - 1) - B_k C_k$, the Jacobian determinant. Therefore, Newton's method gives the following recurrence formula in order to yield a periodic point of F^k.

$$x_{n+1} = x_n - \frac{(D_k - 1)(\overline{x_n} - x_n) - B_k(\overline{y_n} - y_n)}{\Delta}$$

$$y_{n+1} = y_n - \frac{(-C_k)(\overline{x_n} - x_n) + (A_k - 1)(\overline{y_n} - y_n)}{\Delta}$$

$$\textit{where} \quad F^k(\overline{x_n}) = (x_n, y_n)$$

B. Numerical techniques for finding bifurcation values

Let (x_1, y_1), (x_2,y_2),…. (x_k, y_k) be the periodic points of $f(x, y)$ at $a = a_1$. Let λ_1, λ_2 be the two eigen values of J_k at a_1, then (x_i,y_i),i=1,2,…,k are stable if $\max\{|\lambda_1|, |\lambda_2|\} < 1$. It has been observed that $\max\{\lambda_1(a), \lambda_2(a)\}$ is a decreasing function [4]. Let $\lambda(k, a_1)$= min $\{\lambda_1(a_1), \lambda_2(a_1)\}$, where $n=2^k$ is the period number. Then we may apply some of the numerical techniques viz. Bisection method or Regula Falsi method to get the value of 'a' such that $\lambda(k, a) = -1$.

Our numerical results are as follows:

Table 1: Bifurcation points

Sl No.	Bifurcation Point	One of the periodic points
1	10.00000002667134	{0.99999999157105,0.99999999157105}
22 2	13.56082195269039	{0.10953839723577,1.00000000000000}
3	13.86566899263189	{0.17062836281848,0.92065506432418}
4	13.92882843160121	{0.14713139284922,0.94260742121517}
5	13.94222855978345	{0.20063409295004,0.89213212315452}
6	13.94509810925305	{0.15999248423609,0.92990383359027}
7	13.94571260826141	{0.19346405448073,0.89892211747421}
8	13.94584421438873	{0.20204647634563,0.89085678547102}
9	13.94587240029845	{0.20175351752830,0.89115213081720}
10	13.94587843685425	{0.20205781249125,0.89084486037143}

Using the formula $\delta_n = \frac{A_n - A_{n-1}}{A_{n+1} - A_n}$, where A_n represents n^{th} bifurcation point, the Feigenbaum universal constant is calculated with the help of experimentally calculated bifurcation point.

The values of δ_n are as follows.
δ_1 =11.680683947931243, δ_2 =4.82662678637139, δ_3 =4.7133 45880752413, δ_4 =4.669767266324282, $\delta_5 = 4.6697381616044495$, δ_6 =4.669227952149593 & so on.

It is clear that the map obeys Feigenbaum universal behaviour as the sequence $\{\delta_n\}$ converges to δ as n becomes very large.

III. ACCUMULATION POINT

We can calculate the accumulation point using the formula $A_\infty = \frac{A_2 - A_1}{\delta - 1}$, where $\delta = 4.669201609102990671853204$ is Feigenbaum constant. But it has been observed that $\{\delta_n\}$ converges to δ as $n \to \infty$. Hence a sequence of accumulation point $\{A_{\infty,n}\}$ is made using the formula $A_{\infty,n} = \frac{A_{n+1} - A_n}{\delta - 1}$ [11]. From the above experimental values of bifurcation points, the sequence of accumulation points is constructed and it is observed that the sequence converges to the point 13.945880082050246……. After which chaotic region starts.

IV. LYAPUNOV EXPONENTS

The spectrum of Lyapunov exponents is the most precise tool for identification of the character of motion of a dynamical system and its estimation is one of the fundamental tasks in studies of these systems. These exponents are an exponential measure of divergence or convergence of nearby orbits in the phase space. From a mathematical point of view Lyapunov exponents are numbers describing the behaviour of the derivative of transformation along the phase trajectory. In practice, these exponents are a measure of sensitive dependence on initial conditions in phase space. For practical applications it is most important to know the largest Lyapunov exponent. If the largest value in the spectrum of Lyapunov exponents is positive it means that the system is chaotic. The largest value equal to zero indicates periodic system dynamics. If all Lyapunov exponents are negative, then the stable critical point is an attractor. [5,19]

Let us consider a two dimensional difference equation $x_{n+1} = f(x_n, y_n), y_{n+1} = g(x_n, y_n)$ which is depending on a parameter a (say) with initial point (x_0, y_0). Let the evolved iterated points be $(x_1, y_1), (x_2, y_2) .... (x_n, y_n)$ at the parameter $a = a_0$. Suppose that the trajectory moves to the fixed point (x^*, y^*). Since the fixed point is stable and doesn't move under iteration so after finite iterations the trajectory comes very close to the fixed point. Let the Jacobian matrix J^* at the fixed point (x^*, y^*) of the two dimensional map is $J^* = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x^*, y^*)}$. Let the eigenvalue of the J^* at the fixed point (x^*, y^*) be λ_1, λ_2. Now for the nth iterated point (x_n, y_n). The Jacobian matrix (J_n)

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x_n, y_n)} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x_{n-1}, y_{n-1})} \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x_{n-2}, y_{n-2})} K\,K \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x_0, y_0)}$$

i.e. $J_n = J_{n-1}.J_{n-2}.K\, J_0$, as n is sufficiently large the Jacobian matrix will be of the form $J = J^*.J^*.J^* K\, J_{n-1}.J_{n-2}.K\, J_0$, where J^* is the Jacobian matrix at the fixed point (x^*, y^*). So for large iterative value, J^* will gover n and we may neglect the finite number of J_n. Therefore for the large n , the eigen values of J are $\lambda_1{}^n, \lambda_2{}^n$ and then

$$lyapunov\,exponent(\lambda) = \frac{\log(\text{eigen value})}{n}$$

Out of the two lyapunov exponents at every parameter the maximum will be considered which is crucial to detect the dynamic behaviour of the system.

Table 2. Lyapunov exponent near the accumulation point:

Parameter Value	Lyapunov exponent
13.94	-0.00459264
13.945	-0.00233258
13.94588	-0.000358386
13.94588008	-0.000216403
13.94588008205	0.0000228303

Table:3 Lyapunov Exponent at the bifurcation Points:

Parameter Value	Lyapunov exponent
10.00000002667134	-2.57559×10^{-6}
13.56082195269039	4.93845×10^{-6}
13.86566899263189	9.54063×10^{-6}
13.92882843160121	8.71575×10^{-6}
13.94222855978345	9.54063×10^{-6}
13.94509810925305	-0.0000230201
13.94571260826141	-0.0000186148
13.94584421438873	-0.0000367978
13.94587240029845	-9.37354×10^{-6}
13.94587843685425	-2.2895×10^{-6}

Below we have shown the graph of lyapunov exponent versus the parameter value beween 13.5 to 13.96. The graph also shows that almost after the value 13.94588008 of 'a' the remaining lyapunov exponents become positive, showing the beginning of chaotic region. The figure further supports the bifurcation points where the lyapunov exponent is almost zero. The values are also supported by the bifurcation diagram.

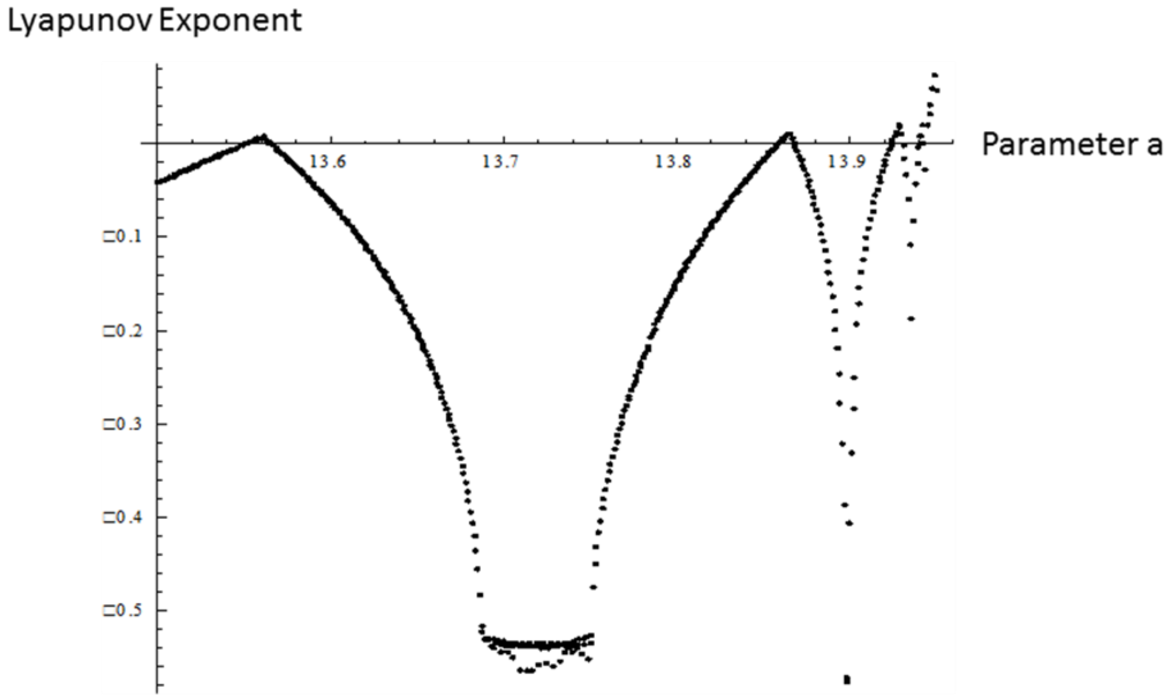

Figure 2: Graph of calculated Lyapunov exponents versus the parameter a **for** $13.5 \le a \le 13.96$

V. NUMERICAL METHODS FOR CALCULATING FRACTAL DIMENSIONS OF THE MAP

Fractals arise from a variety of sources and have been observed in nature and on computer screens. One of the exceptional characteristics of fractals is that they can be described by a non integer dimension. The geometry of fractals and the mathematics of fractal dimension have provided useful tools for a variety of scientific disciplines, among which is chaos. Chaotic dynamical systems exhibit trajectories in their phase space that converge to a strange attractor. The fractal dimension of this attractor counts the effective number of degrees of freedom in the dynamical system and thus quantifies its complexity [22].

A. Box-counting dimension

The box-counting dimension is motivated by the notion of determining space filling properties of a curve. In this approach, the curve is covered with a collection of area elements (square boxes), and the number of elements of a given size is counted to see how many of them are necessary to cover the curve completely. As the size of the area element approaches zero, the total area covered by the area elements will converge to the measure of the curve. This can be expressed mathematically as [8, 16]

$$D_B = \lim_{r\to 0} \frac{\log N(r)}{\log\left(\frac{1}{r}\right)}$$

where $N(r)$ is the total number of boxes of size r required to cover the curve entirely. However in practice, the box-counting algorithm estimates fractal dimension of the curve by counting the number of boxes required to cover the curve for several box sizes, and fitting a straight line to the log-log plot of $N(r)$ versus r . The slope of the least square best fit straight line is taken as an estimate of the box-counting dimension D_B of the curve. This procedure is also called grid
method and involves two dimensional processing of the curve at multiple grid sizes, which is computationally highly time consuming.

B. Information dimension

As with the box-counting dimension, the attractor is covered with hypercubes of side length.

This time, however, instead of simply counting each cube which contains part of the attractor, we want to know how much of the attractor is contained within each cube. This measure seeks to account for differences in the distribution density of points covering the attractor, and is defined as [1, 8]

$$D_I = \lim_{r\to 0} \frac{I(r)}{\log\left(\frac{1}{r}\right)}$$

, where is given by Shannon's entropy formula, $I(r) = -\sum_{i=1}^{N} P_i \log(P_i)$, where P_i is the probability of part of the attractor occurring within the ith hypercube of side length r .For the special case of an attractor with an even distribution of points, an identical probability , $P_i = \frac{1}{N}$ is associated with every box. Hence, $I(r) = \ln(N)$.Consequently, $B_b = B_I$. Thus, D_B simply counts all hypercubes containing parts of the attractor, whereas D_I asks how much of the attractor is within each hypercube and correspondingly weights its count.

C. Correlation dimension

Correlation dimension describes the measure of dimensionality of the space occupied by chaotic attractor of any system having presence of complexity.

During numerical simulation, correlation dimension can be calculated using the distances between each pair of points in the set of *N* number of points, $s_{ij} = | X_i - X_j |$. To calculate the correlation dimension [8,17] the following steps must be followed :

For an orbit $O(X_1) = \{X_1, X_2, X_3, X_4\}$ of a map $f : U \to U$, where U is an open bounded set in P^n and for a given positive real number *r*, first we obtain the correlation integral,

$$C(r) = \lim_{n\to\infty} \frac{1}{n(n-1)} \sum_{i\neq j}^{N} \Theta\left(r - \| X_i - X_j \|\right)$$

Where Θ is the unit-step function, (Heaviside function). The summation indicates that the number of pairs of vectors closer to *r* for $1 \le i, j \le n$ and $i \neq j$. The *C*(*r*) measures the density of a pair of distinct vectors X_i and X_j that are closer to r. Then, the correlation dimension D_c of $O(X_1)$ is defined as

$$D_c = \lim_{r\to 0} \frac{\log C(r)}{\log(r)}$$

To obtain D_c , log *C*(*r*) is plotted against log *r* and then we find a straight line fitted to this curve. The y intercept of this straight line provides the value of the correlation dimension D_c .

D. GENERALISED CORRELATION DIMENSION

Dissipative dynamical systems which exhibit chaotic behaviour often have an attractor in phase space which is strange. Strange attractors are typically characterized by fractal dimensionality. For this purpose, different kind of dimensions for the strange attractor of the map (1) have been studied.

At the onset of chaos, i.e. at the accumulation point, the dimension of the attractor is measured. For that, information dimension and box counting dimension are studied as well.

The help of generalized correlation sum is taken, to calculate these dimensions. First of all let discuss generalized dimension **[7, 10, 20]**, i.e. Reni's dimension. It is defined as follows:

$$D_q = \frac{\lim}{R \to 0} \frac{1}{q-1} \frac{\log \sum_{i=1}^{N(R)} P_i^q}{\log R} \quad \text{........ (2)}$$

Where P_i is the probability and is defined as $P_i = N_i / N$ and R is the side length.

If q = 0, it gives the box counting dimension. For q=1, it gives information dimension and for q=2 it gives correlation dimension.

Again, generalized correlation sum is given as follows:

$$G_q (N,R) = \left[\frac{1}{N-1} \sum_{i=1}^{N} \left[\frac{1}{N-1} \sum_{j=1, j\neq i}^{N} H(R - \|X_i - X_j\|) \right]^{q-1} \right]^{\frac{1}{q-1}} \quad \text{......... (3)}$$

Where $\mathbf{H}\left(\mathbf{R} - \|\mathbf{X_i} - \mathbf{X_j}\|\right)$ means if $\|\mathbf{X_i} - \mathbf{X_j}\| < R$, then $\mathbf{H}\left(\mathbf{R} - \|\mathbf{X_i} - \mathbf{X_j}\|\right) = 1$ and

if $\|\mathbf{X_i} - \mathbf{X_j}\| \ge R$, then $\mathbf{H}\left(\mathbf{R} - \|\mathbf{X_i} - \mathbf{X_j}\|\right) = 0$.

Further, $\frac{\lim}{\mathbf{N} \to \infty} G_q (N,R) = G_q (R)$, and

$$D_q = \frac{\lim}{\mathbf{R} \to 0} \frac{\log \mathbf{G_q(R)}}{\log \mathbf{R}} \quad \text{......... (4)}$$

Thus, if G_q (N, R) is known, the value of G_q(R) can be found easily. And hence D_q can be obtained for a particular value of q. For this purpose , calculate G_q(30000,r) for different values of q and consider that as G_q(r). However it will be justified due to the convergence nature of G_q(N,R) as N becomes large. We consider the range of R as 10^{-7} to 10^{-1} and get the value of log G_q(R).Then plot (log R, log G_q(R)) out of which scaling region is selected. In the scaling region we fit a straight line whose slope will give the dimension.

To apply (3) we use the Euclidean norm i.e. if $X_1=(x_1,y_1)$ and $X_2=(x_2,y_2)$ then

$$||X_1-X_2||=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

1) Correlation dimension

$G_q(R)=G_q(30000,R)$ is calculated . The part of the plotted points (log R, log G_q(R)) which follows equation (4) is taken . The slope of the fitted straight line in that scaling region is D_q. Calculated value of (log R, log G_q(R)) in the scaling region are as follows:

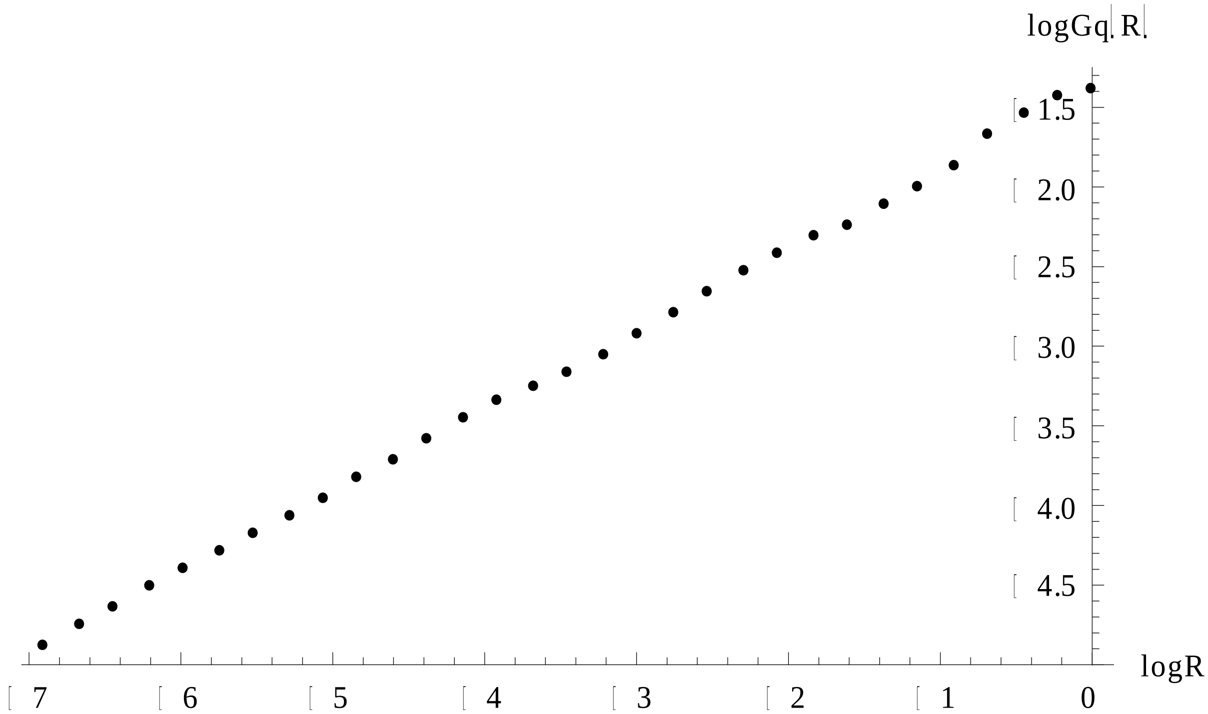

Figure 3: log R vs. log G_q(R) in the scaling region

The slope of the above points when fitted with a straight line by least square method is 0.50684 with a mean deviation of 0.0629964. The data is obtained from 30000 iterated points at the parameter13.945880082050246**.**

2) Information dimension

If we take q very near to one i.e. q tends to 1 then that gives the information dimension. At q=1.00000000001 at the parameter 13.945880082050246with 30000 iterations.

Calculated value of (log R, log G_q(R)) in the scaling region are as follows:

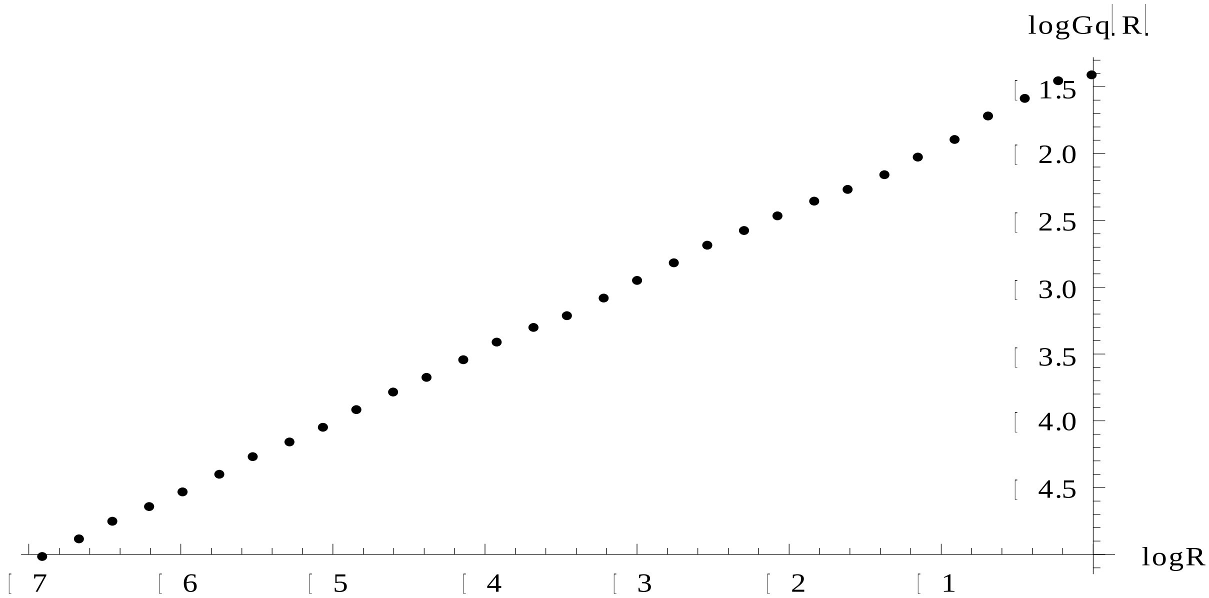

Figure 4: log R vs. log G_q(R) in the scaling region

The slope of the above points when fitted with a straight line by least square method is 0.523461with a mean deviation of 0.0573148.

3) Box counting dimension

For q =0 it gives the box counting dimension. We have done the calculation at the parameter 13.945880082050246with 30000 iterations. Calculated value of (log R, log G_q(R)) in the scaling region are as follows:

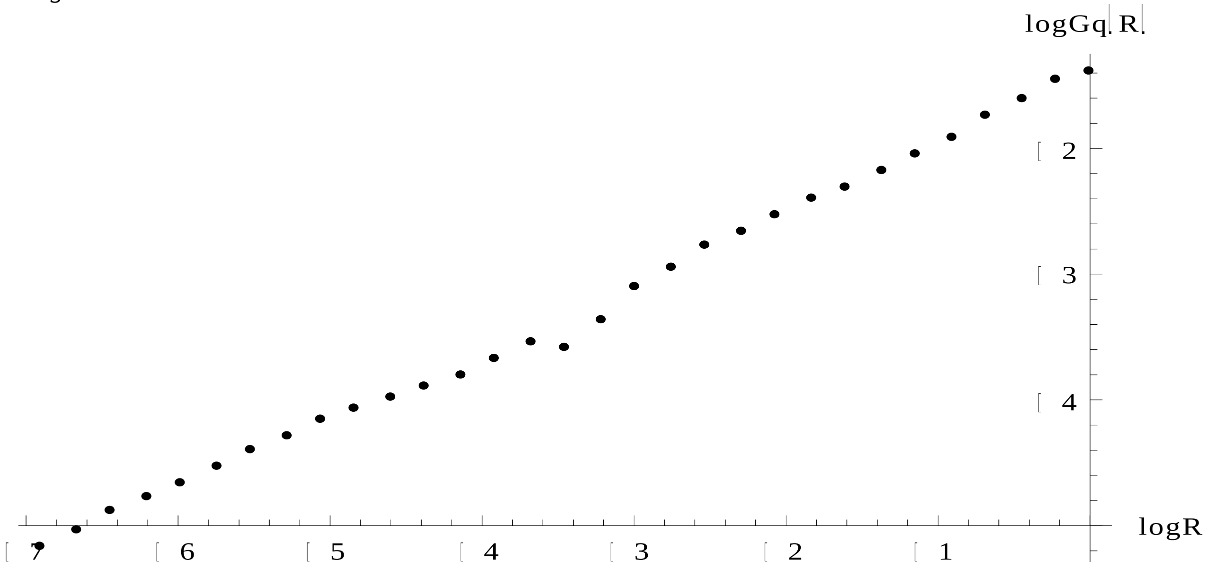

Figure 5: log R vs. log G_q(R) in the scaling region

The slope of the above points when fitted with a straight line by least square method is 0.548454with a mean deviation of 0.130303.

VI. CONCLUSION

The numerical techniques used in this paper to calculate Lyapunov exponent and other fractal dimensions may be used to calculate the same in higher dimensional models.

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