International Journal of Scientific and Research Publications, Volume 4, Issue 5, May 2014 1
ISSN 2250-3153

Extra and Alternative Loop Rings

K.Jayalakshmi[1] and C.Manjula[2]

[1]Assistant Professor in Mathematics, J.N.T.University Anantapur College of Engg., J.N.T.University Anantapur. Anantapur.(A.P) INDIA.
[2] Department of Mathematics, J.N.T.University Anantapur College of Engg., J.N.T.University Anantapur. Anantapur.(A.P) INDIA.

Abstract- The right alternative law implies the left alternative law in loop rings of characteristic other than 2. We show that there exists a loop which fails to be an extra loop, even though its characteristic 2 loop rings are right alternative.

Index Terms- Alternative loop, extra loop, inverse property, alternative ring, alternative loop ring.

2010 MATHEMATICS SUBJECT CLASSIFICATION: 17D05, 17D15.

I. INTRODUCTION

Throughout this paper $(L, \cdot)$ always denotes a loop, with identity element e and $(R, +, \cdot)$ always denotes an nonassociative commutative ring, with identity element $1 \neq 0$. Then RL denotes the loop ring constructed from R and L. Elements of RL are represented by finite formal sums of the form $\sum_{i<n} a_i\ x_i$, where the x_i are elements of L and the a_i are elements of R. The sum and product operations on RL are defined in the obvious way. Then $1e$ is the identity element of RL. For more details, background information, and references to the earlier literature see the survey by Goodaire and Milies [1]. We note that L is embedded in to RL via the map $x \rightarrow 1x$; we usually write the element $1x$ simply as x. In this paper we would like to focus on right alternative law $((x\ y)y = x(y\ y))$ and the left alternative law $((y\ y)x = y(y\ x))$. The right alternative law in RL is precisely more efficient over the right alternative law in L.

In the case that L is finite; Theorem 2 was proved by Chein and Goodaire [7]. Their proof relied on structure theorems of Bruck and Albert which are false for infinite L. Goodaire and Robinson [2 and 3] showed that the assumption in the theorem that R satisfies $1 + 1 \neq 0$ cannot be dropped.
Actually, in the study of alternative laws in loop rings RL, only L is relevant, by the following result from [7]:

REMARK : RL is associative iff L is associative.

THEOREM 1: A loop L is an RA loop if and only if L is not associative and has the following properties:
i) if three elements of L associative in some order, then they associate in all orders;
ii) if $g, h, k \in L$ do not associate, then $g \cdot kh = gh \cdot k = h \cdot gk$.

PROOF : First, suppose that L is an RA loop and R is some ring of characteristic different from 2. Since $L \subseteq RL$ and RL is an alternative ring, statement (i) holds by the Generalized Theorem of Artin [5]. To obtain (ii) let g, h and k be three elements of L which do not associate and put $x = h + k$, $y = g$ in the right alternative identity, $yx \cdot x = yx^2$. We obtain $gh \cdot h + gh \cdot k + gk \cdot h + gk \cdot k = gh^2 + g \cdot hk + g \cdot kh + gk^2$ and since $gh \cdot h = gh^2$ and $gk \cdot k = gk^2$, $gh \cdot k +$
$gk \cdot h = g \cdot hk + g \cdot kh$. Because char $R \neq 2$, $gh \cdot k$ is in the support of the left side and thus in the support of the right. So, if $gh \cdot k \neq g \cdot hk$, then $gh \cdot k = g \cdot kh$. Similarly, by considering the left alternative identity, we obtain $gh \cdot k =$
$h \cdot gk$.

Conversely, suppose that L is a loop which satisfies statements (i) and (ii) of the theorem. Let R be a commutative and associative ring with unity. For $x = \sum\alpha_g\, g$ and $y = \beta_g\, g$ in RL, $yx \cdot x - yx^2$ is a linear combination of terms of the form $gh \cdot k - g \cdot hk$. If $h = k$, then $gh \cdot h = g\ .\ hh$ by associativity or (ii), so $yx \cdot x - yx^2$ reduces to a sum of terms of the form $(gh \cdot k - g \cdot hk) + (gk \cdot h - g \cdot kh)$ with $h \neq k$ and this is 0 by (i) and (ii). This shows that the right alternative identity holds in RL. The left alternative identity follows in a similar way.

COROLLARY 1: An RA loop is an extra loop.

PROOF: Let L be an RA loop. If x, y and z are three elements of L which associative, then, by Moufang's Theorem, these elements generate a group, so $(xy \cdot z)x = x(y \cdot zx)$ is clear. If x, y and z do not associative, then neither do
$(xy \cdot z)x = x(y \cdot zx)$ is clear. If x, y and z do not associative, then neither do xy, z, x (again by Moufang's Theoram), so $(xy \cdot z)x = (x\ y)(x\ z)$ by Theorem 1 and, since x, y, xz do not associate, $(x\ y)\ (x\ z) = x(xz \cdot y)$. Repeated use of Theorem 1 gives $xz \cdot y = z \cdot xy = zy \cdot x = y \cdot zx$ and so, again $(xy \cdot z)x = x(y \cdot zx)$.

We begin by eliminating the rings from the theory of loop rings. Lemma 1 almost does that, since it expresses right alternativity in RL just in terms of elements of the form $1x$ (which, recall, we are writing as x). The material through Lemma 2 is from Chein and Goodaire [7].
LEMMA 1 : RL is right alternative iff L is right alternative and RL satisfies

$x(y\,z) + x(z\,y) = (x\,y)z + (x\,z)\,y$ for all $x, y, z \in L$.

PROOF : If RL is right alternative, then L is trivially also right alternative, but also RL must satisfy $x\,(y\,y) = (x\,y)\,y$ for any $x, y \in RL$. Linearization of this gives $x\,(y\,z) + x\,(z\,y) = (x\,y)\,z + (x\,z)\,y$. Conversely, assuming this equation and the right alternativity of L, it is easy to verify $x\,(y\,y) = (x\,y)\,y$. □

Now, if p, q, r, s are arbitrary elements of L, the equation $p + q = r + s$ cannot hold in RL unless $p = r$ and $q = s$ or $p = s$ and $q = r$, except in the case that R has characteristic 2, in which case there is also the possibility that $p = q$ and $r = s$. Applying this observation to the result of the above Lemma 1, the alternative law in RL reduces to a Boolean combination of equations in L as follows:

DEFINITION: In any loop, define the properties $A(x, y, z)$, $B(x, y, z)$ and $C(x, y, z)$ by:
$A(x, y, z)$ is $x\,(z\,y) = (x\,z)\,y$.
$B(x, y, z)$ is $x\,(z\,y) = (x\,y)\,z$.
$C(x, y, z)$ is $(x\,y)\,z = (x\,z)\,y$.

LEMMA 2: For any R and L:
If $1 + 1 \neq 0$ in R then RL is right alternative iff for all x, y, z in L, either $A(x, y, z)$ or $B(x, y, z)$ holds.
If $1 + 1 = 0$ in R then RL is right alternative iff L is right alternative and for all x, y, z in L either $A(x, y, z)$ or $B(x, y, z)$ or $C(xy, y, z)$ holds.

PROPOSITION 1: For any L:
RL satisfies the right alternative law for some R of characteristic = 2 iff RL satisfies the right alternative law for all R of characteristic = 2.
RL satisfies the right alternative law for some R of characteristic ≠ 2 iff RL satisfies the right alternative law for all R of characteristic ≠ 2.
This follows immediately from the fact that one can express the right alternativity of RL by a Boolean combination of equations in L; there is one Boolean combination for the characteristic = 2 case and another for the characteristic ≠ 2 case; see [7] and Lemma 2 above.
Proposition 1 is immediate from Lemma 2. Note that both $A(x, y, y)$ and $B(x, y, y)$ reduce to $x\,(y\,y) = (x\,y)\,y$, so that we do not have to postulate the right alternativity of L in the characteristic ≠ 2 case.

LEMMA 3: For any R of characteristic other than 2 and any L, RL is right alternative iff for all x, y, z in L, we have:
1. $x(y\,z) = (x\,z)y$
2. $x(z\,y) = (x\,y)z$

PROOF : Write $A(x, y, z)$ as $P_1(x, y, z)$ and $B(x, y, z)$ as $Q_1(x, y, z)$. Then by the previous Lemma, RL is right alternative iff for $i, j = 1, 2$ we have $P_i\,(x, y, z)$ or $Q_j\,(x, y, z)$ for each $x, y, z \in L$. But, by remaining the variables, these two statements reduce to just (1) and (2). □

We turn now to the proof of theorem 2. If L is a loop with product '·', let L^{op} denote the opposite loop, $(L, \circ)$, defined by defined by $x \circ y = y \cdot x$. But, if R is a non-associative ring, we let R^{op} denote the ring $(R,+)$; R and R^{op} have the same + operation. Now, in forming our loop rings, R was always commutative, so $(RL)^{op} \cong R(L^{op})$. Clearly, RL is left alternative iff $(RL)^{op}$ is right alternative, so we shall be done if we can show that $RL \cong (RL)^{op}$, which in turn will form $L \cong L^{op}$.

To prove $L \cong L^{op}$, denote an inverse map, $i(x)$ by $x \cdot i(x) = e$; equivalently, $i(x) = x/e$. Just by the loop properties, i is a bijection from L on to L. We shall prove that $i(x \cdot y) = i(y) \cdot i(x)$, so that i is an isomorphism from L on to L^{op}. In hindsight, this is not surprising, since the theorem implies that L is Moufang and hence satisfies the inverse property. So, we proceed with a few Lemmas about $i(x)$.

LEMMA 4: If RL is right alternative and R has characteristic other than 2, then $i(x) \cdot x = e$ for all $x \in L$.

PROOF : Fix a and let $b = i(a)$ so that $a\,b = e$ and assume $b\,a \neq e$. Then fix c such that $c\,a = e$; so $b \neq c$. We shall derive a contradiction by using (1) and (2) of Lemma 3.
First, we have $c\,(a\,b) = c \neq b = (c\,a)\,b$. Applying (1) we get
$c\,(a\,b) = (c\,b)\,a$, so $(c\,b)\,a = c$ … (i)
Applying (2) we get $c\,(b\,a) = (c\,a)\,b$, so $c\,(b\,a) = b$ … (ii)
Applying (i) and the right alternative law $(c\,b)\,a^2 = e$ … (iii)
Applying (1), we have $c\,(b\,a^2) = (c\,a^2)\,b$. But by (iii), right alternativity and the definitions of b and c, both equations simplify to $c\,(b\,a^2) = e$ … (iv)
Applying (1) again $c\,((b\,a)\,a) = (c\,a)(b\,a)$. But by right alternativity, (iv), (ii) and the definition of both equations simplify to $b\,a = e$, a contradiction.
So, we have $i(x) \cdot x = x = x \cdot i(x) = e$, which immediately implies $i\,(i(x)) = x$. □

LEMMA 5: If RL is right alternative and R has characteristic other than 2, then $(y \cdot i(x)) \cdot x = y$ for all $x, y \in L$.

PROOF : Apply (2) of Lemma 3 to get $y \cdot (x \cdot i(x)) = (y \cdot i(x)) \cdot x$ this implies $(y \cdot i(x)) \cdot x = y$. □

LEMMA 6: If RL is right alternative and R has characteristic other than 2, then $i(x) \cdot (x \cdot y) = y$ for all $x, y \in L$.

PROOF : Fix any $a, b \in L$ and let $\hat{a} = i(a)$ so $a\,\hat{a} = \hat{a}\,a = e$. We assume $\hat{a}(a\,b) \neq b$ and derive a contradiction. Applying (2) of Lemma 3 to get $\hat{a}(b\,a) = (\hat{a}\,a)b$ which implies $\hat{a}(b\,a) = b$ (since $\hat{a}(a\,b) \neq b$). Applying (2) again, $a((b\,a)\,\hat{a}) = (a\,\hat{a})(b\,a)$. Using $\hat{a}(b\,a) = b$ and Lemma 5, both these equations reduce to $b\,a = a\,b$, so that we have $\hat{a}(a\,b) = b$, a contradiction. □

THEOREM 2 : Suppose that RL satisfies the right alternative law and R satisfies $1 + 1 \neq 0$. Then RL (and hence also L) satisfies the left alternative law.

PROOF : For any $x, y \in L$, we have by applying Lemmas 6, 5, 5 in that order, $i(x\,y) \cdot [(x\,y) \cdot i(y)] = i(y)$ and then $i(x\,y) \cdot x = i(y)$, and then $i(x \cdot y) = i(y) \cdot i(x)$ which, as remarked above, is sufficient to prove the theorem. □

EXTRA LOOPS

As pointed out in the introduction, Goodaire and Robinson [2 and 3] showed that Theorem 2 can fail if R has characteristic 2. Their examples all satisfied the right extra identity, $(xy \cdot z)x = x(y \cdot zx)$ and they ask whether this is necessary. That may seem plausible, since if RL is both left and right alternative, then regardless of the characteristic, L satisfies the Moufang identities, which imply the Extra identity. However, it turns out that right alternativity alone of RL does not even imply the special case of the Extra identity when $x = y = z$ - namely, $x^2\,x = x\,x^3$. Note that right alternativity does imply that $x^2\,x = x\,x^2$, so the notation x^3 is unambiguous. Note also that left alternativity of L then fails in our example, since otherwise $x^3\,x = (x^2\,x)\,x = x^2\,x^2 = x\,(x\,x^2) = x\,x^3$.

THEOREM 3: For each $n \geq 3$, there is a loop L of size $2n$ such that RL is right alternative whenever R has characteristic 2, but L does not satisfy $x^3\,x = x\,x^3$.

PROOF : As a set, let L be $\{j : 0 \leq j < 2n\}$. On this set + will always denote addition modulo $2n$. Let φ be a permutation of the set of odd elements, $\{2i + 1 : 0 \leq I < n\}$. Given φ we define the operation '$\circ$' on L by letting $x \circ y$ be $x + y$ unless x, y are both odd, in which case we let $x \circ y = x + \varphi(y)$. We shall show that for some choices of φ, $(L, \circ)$ satisfies the theorem.

First, using the fact that φ is a permutation, it is easy to see that $(L, \circ)$ is a loop, with identity element 0.

Next, note that L is right alternative, since for odd y, we have $x \circ (y \circ y) = x + y + \varphi(y) = (x \circ y) \circ y$, while for even y, we have $x \circ (y \circ y) = x + y + y = (x \circ y) \circ y$.

Whenever x is odd $x^3\,x = 2x + 2\,\varphi(x)$, while $x\,x^3 = x + \varphi(2x + \varphi(x))$. We can make these differ for $x = 1$ by letting $\varphi(1) = 1$ and $\varphi(3) \neq 3$.

Finally, to prove RL is right alternative, we apply Lemma 2 and show and that at least one of $A(x, y, z)$, $B(x, y, z)$, $C(x, y, z)$ holds for $x, y, z \in L$. We consider the possible cases for x, y, z.

If at least two of x, y, z are even, then all possible associations and commutations of $x \circ y \circ z$ evaluate to $x + y + z$, so that $A(x, y, z)$, $B(x, y, z)$, $C(x, y, z)$ all hold.

If x, y, z are all odd, then $x \circ (y \circ z) = x + y + \varphi(z) = (x \circ z) \circ y$ and $x \circ (z \circ y) = x + z + \varphi(y) = (x \circ y) \circ z$, so that $B(x, y, z)$ holds.

If x is even and y, z are odd, then $x \circ (y \circ z) = x + y + \varphi(z) = (x \circ y) \circ z$ and $x \circ (z \circ y) = x + z + \varphi(y) = (x \circ z) \circ y$ so that $A(x, y, z)$ holds.

If y is even and x, z are odd, then $x \circ (y \circ z) = x + \varphi(y + z) = x \circ (z \circ y)$ and $(x \circ y) \circ z = x + y + \varphi(z) = (x \circ z) \circ y$, so that $C(x, y, z)$ holds. Likewise, $C(x, y, z)$ holds in the remaining case, where z is even and x, y, z are odd.

REFERENCES

[1] E. G. Goodaire and C. P. Milies, "Ring alternative loops and their loop rings", Resenhas 2 (1995) 47-82.

[2] E.G. Goodaire and D. A. Robinson, "A class of loops with right alternative loop rings", Communications in Algebra 22 (1994) 5623-5634.

[3] E. G. Goodaire and D.A. Robinson, "A construction of loops which admit right alternative loop Rings", Results in mathematics 29 (1996) 56-62.

[4] F. Fenyves, "Extra loops. I", Publ.Math.debrecen 15 (1968), 235-238.MR 38 #5976.

[5] K.A. Zhevlakov, A.M. Slin'ko, I.P. Shestakov, and A. Shirshov, "Rings that are nearly Associative", Academic Press, New York, 1982. Associative", Academic Press, New York, 1982.

[6] O. Chein and D. A. Robinson, "An extra law for characterizing Moufang loops", Proceedings of the American Mathematical society, vol.33, No.1, May 1972.

[7] O. Chein and E. G. Goodaire, "Is a right alternative loop ring alternative", Algebra Groups Geom. 5 (1988) 297-304.

AUTHORS

First Author – K. Jayalakshmi, Assistant Professor in Mathematics, J.N.T.University Anantapur College of Engg. J.N.T.University Anantapur, Anantapur. (A.P) INDIA. jayalakshmikaramsi@gmail.com

Second Author – C. Manjula, Department of Mathematics, J.N.T.University Anantapur College of Engg. J.N.T.University Anantapur, Anantapur. (A.P) INDIA. man7ju@gmail.com