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Application of Salagean differential operator to certain Subclasses of harmonic univalent functions

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Abstract: For analytic functions of the form $f_i(z) = z + \sum_{n=2}^{\infty} a_n^i z^n$ in the open unit disk, we define the Salagean differential operator $D^n f_i(z)$ to be

$$D^n f_i(z) = z + \sum_{n=2}^{\infty} n^k a_n^i z^n$$

In this paper, we investigated some properties of $D^n f_i(z)$ for $\Gamma_\alpha^n(\zeta_{1,}\zeta_2;\gamma)$ in

$$F_\alpha(z) = \int_0^z \prod_{i=1}^k \left(\frac{D^n f_i(s)}{s}\right)^{1/\alpha} ds, \alpha \epsilon C\ |\alpha| \leq 1$$

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Index Terms: Analytic, Univalence, Harmonic, Coefficient bound, Convolution, Convex combination.

I. INTRODUCTION

Let Hdenote the family of continuous complex-valued functions which are harmonic in the open unit disk $U = \{z : |z| < 1\}$ and let A be the subclass of H consisting of functions which are analytic in U. A function harmonic in U may be written as
$f = h + \bar{g}$, where h is refered to as the analytic part and g the co-analytic part of f and h, g are members of A. f is said to be sense preserving if $|h'(z)| > |g'(z)|$ where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} a_n z^n \qquad (1)$$

It is observed that the sense preserving property implies that $|b_1| < 1$. Let SH denote the family of functions $f = h + \bar{g}$ which are harmonic univalent and sense-preserving in
U for which f is normalized with $f_z(0) = f'_z(0) - 1 = 0$. It noted that SH reduces to the class S of normalized analytic univalent function in U if the co-analytic part of f
is identically zero. Clunie and Sheil-Small [3] in 1984 investigated the class SH as well
as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on SH and its subclasses. Elif Yasar and Sibel Yalcni
[7] studied convolution properties of Salagean-type of harmonic univalent functions,
Ahuja and Jahangiri[2], Makinde [5] studied convolution of special class of harmonic
univalent functions. N.Seenivasagan [6] gave a condition for the univalence of the integral operator defined by

$$F_{\alpha,\beta}(z) = \left\{\beta \int_0^z t^{\beta-1} \prod_{i=1}^k \left(\frac{f_i(t)}{t}\right)^{1/\alpha} dt\right\}^{1/\alpha}$$

Where $f_i(z\ )$ is defined as

$$f_i(z) = z + \sum_{n=2}^{\infty} a_n^i z^n \tag{2}$$

The differential operator $D^n\ (n \epsilon N)_0$ was introduced by Salagean [6]. Where $D^n f(z)$ is defined by

$$D^n f(z) = D\left(D^{n-1} f(z)\right) = z\left(D^{n-1} f(z)\right)' with\ D^0 f(z) = f(z) \tag{3}$$

We give the Salagean Differential operator for the function $f_i(z)$ in (2) to be of the form

$$D^n f_i(z) = z + \sum_{n=2}^{\infty} n^k a_n^i z^n \tag{4}$$

Let $f_i(z) = h_i(z) + \bar{g}_i(z)$ where

$$h_i(z) = z + \sum_{n=2}^{\infty} a_n^i z^n, \qquad g_i(z) = \sum_{n=1}^{\infty} b_n^i z^n \tag{5}$$

And

$$D^n F_i(z) = z + \sum_{n=2}^{\infty} n^k a_n^i z^n + \overline{\sum_{n=1}^{\infty} n^k b_n^i z^n} \tag{6}$$

$$F_\alpha(z) = \int_0^z \prod_{i=1}^k \left(\frac{D^n f_i(s)}{s}\right)^{1/\alpha} ds, \alpha\ \epsilon\ C\ |\alpha| \leq 1 \tag{7}$$

Where $f_i(z)$ is as in (2). Makinde and Oladipo in [5] investigated some properties of

$\Gamma_\alpha^n(\zeta_{1,}\zeta_2;\gamma)$ for the function $f_i(z)$ in $F_\alpha(z)$ defined by (7).

We define

$$\Gamma_\alpha^n(\zeta_{1,}\zeta_2;\gamma) = \left\{F_i\ \epsilon\ A: \left|\frac{G(z) + \frac{1}{\alpha} - 1}{\zeta_{1,}\left(G(z) + \frac{1}{\alpha}\right) + \zeta_2}\right| \leq \gamma\right\}$$

Where

$$G(z) = \frac{zF'_\alpha(z)}{F_\alpha(z)} = \sum_{i=1}^k \frac{1}{\alpha}\left(\frac{D^{n+1} f_i(z)}{D^n f_i(z)} - 1\right)$$

$$D^n H_i(z) = z + \sum_{n=2}^{\infty} n^k c_n^i z^n + \overline{\sum_{n=1}^{\infty} n^k d_n^i z^n} \tag{8}$$

Moreover, we recall the following definition of the well known classes of starlike and convex functions S^* and S^c given by Acu and Owa [1] denoted by

$$S^* = \left\{f \in A: Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, z \in U\right\} \tag{9}$$

$$S^c = \left\{f \in A: Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, z \in U\right\} \tag{10}$$

We introduce the class of function

$$F_i(z) * D^n H_i(z) = (D^n F_i * D^n H_i)(z) = z + \sum_{n=2}^{\infty} n^k a_n^i\, c_n^i z^n + \overline{\sum_{n=1}^{\infty} n^k b_n^i d_n^i\, z^n} \quad (11)$$

In this paper, we obtain some properties of the class $\Gamma_\alpha^n(\zeta_{1,}\zeta_2;\gamma)$.

II. Main Results

Theorem 1 Let $D^n F_i(z)$ be as in (6) and $F_\alpha(z)$ be as in (7). Then $D^n F_i(z)$ is in the class $\Gamma_\alpha^n(\zeta_{1,}\zeta_2;\gamma)$ if and only if

$$\sum_{i=1}^{k}\sum_{n=2}^{\infty}\{n^k[n(1+\gamma\zeta_{1,}) + \alpha(\gamma\zeta_2 - 1)]\}|a_n^i + b_n^i| \le 2[\gamma|\zeta_1 + \alpha\zeta_2| - |1-\alpha|] \quad (10)$$

$$\zeta_1(0 \le \zeta_1 \le 1), \qquad \zeta_2(0 < \zeta_2 \le 1), \qquad \alpha(0 < \alpha \le 1); \quad |b_n^i| = 1$$

Proof

$$\left|\frac{G(z) + \frac{1}{\alpha} - 1}{\zeta_{1,}\left(G(z) + \frac{1}{\alpha}\right) + \zeta_2}\right| = \left|\frac{\sum_{i=1}^{k} D^{n+1}F_i(z) - \alpha D^n F_i(z)}{\sum_{i=1}^{k} \zeta_1 D^{n+1}F_i(z) - \alpha\zeta_2 D^n F_i(z)}\right|$$

$$= \left|\frac{\sum_{i=1}^{k} 2z(1-\alpha) + \sum_{n=2}^{\infty} n^k(n-\alpha)\left(a_n^i z^n + \overline{b_n^i z^n}\right)}{\sum_{i=1}^{k} 2z(\zeta_1+\alpha\zeta_2) + \sum_{n=2}^{\infty} n^k(n\zeta_1+\alpha\zeta_2)\left(a_n^i z^n + \overline{b_n^i z^n}\right)}\right|$$

$$\le \frac{2|1-\alpha| + \sum_{i=1}^{k}\sum_{n=2}^{\infty} n^k(n-\alpha)|a_n^i + b_n^i|}{2|\zeta_1 + \alpha\zeta_2| - \sum_{i=1}^{k}\sum_{n=2}^{\infty} n^k(n\zeta_1 + \alpha\zeta_2)|a_n^i + b_n^i|}$$

Let $D^n F_i(z)$ satisfies inequality (10), then $D^n F_i(z)$ is in the class $\Gamma_\alpha^n(\zeta_{1,}\zeta_2;\gamma)$.

Conversely, let $D^n F_i(z)$ be in the class $\Gamma_\alpha^n(\zeta_{1,}\zeta_2;\gamma)$ then

$$\sum_{i=1}^{k}\sum_{n=2}^{\infty}\{n^k[n(1+\gamma\zeta_{1,}) + \alpha(\gamma\zeta_2 - 1)]\}|a_n^i + b_n^i| \le 2[\gamma|\zeta_1 + \alpha\zeta_2| - |1-\alpha|]$$

Corollary 1 If $D^n F_i(z)$ is in the class $\Gamma_\alpha^n(\zeta_{1,}\zeta_2;\gamma)$ then

$$|a_n^i + b_n^i| \le \frac{2[\gamma|\zeta_1 + \alpha\zeta_2| - |1-\alpha|]}{n^k[n(1+\gamma\zeta_{1,}) + \alpha(\gamma\zeta_2 - 1)]}$$

Corollary 2 If $D^n F_i(z)$ is in the class $\Gamma_\alpha^n(\zeta_{1,}\zeta_2;\gamma)$ then

$$|a_n^i| \le \frac{2[\gamma|\zeta_1 + \alpha\zeta_2| - |1-\alpha|] - |b_n^i| n^k[n(1+\gamma\zeta_{1,}) + \alpha(\gamma\zeta_2 - 1)]}{n^k[n(1+\gamma\zeta_{1,}) + \alpha(\gamma\zeta_2 - 1)]}$$

Corollary 3 If $D^n F_i(z)$ is in the class $\Gamma_\alpha^n(\zeta_{1,}\zeta_2;\gamma)$ then

$$|b_n^i| \le \frac{2[\gamma|\zeta_1 + \alpha\zeta_2| - |1-\alpha|] - |a_n^i| n^k[n(1+\gamma\zeta_{1,}) + \alpha(\gamma\zeta_2 - 1)]}{n^k[n(1+\gamma\zeta_{1,}) + \alpha(\gamma\zeta_2 - 1)]}$$

Theorem 2 Let $D^n F_i(z)$ be in the class $\Gamma_\alpha^n(\zeta_{1,}\zeta_2;\gamma)$ and the function $D^n H_i(z)$ defined by

$$D^nH_i(z) = z + \sum_{n=2}^{\infty} n^k A_n^i z^n + \overline{\sum_{n=1}^{\infty} n^k B_n^i z^n}$$

Be in the same class $\Gamma_\alpha^n(\zeta_{1,}\zeta_2;\gamma)$. Then the function H$(z)$ defined by

$$\mathrm{H}(z) = (1-\lambda)D^nF_i(z) + \lambda\, D^nH_i(z) = z + \sum_{n=2}^{\infty} n^k C_n^i z^n$$

Is also in the class $\Gamma_\alpha^n(\zeta_{1,}\zeta_2;\gamma)$, where

$C_n^i = (1-\lambda)\left(a_n^i + \overline{b_n^i}\right) + \lambda\left(A_n^i + \overline{B_n^i}\right)$

Proof: Suppose that each of $D^nF_i(z)$, $D^nH_i(z)$ is in the class $\Gamma_\alpha^n(\zeta_{1,}\zeta_2;\gamma)$.

Then by(10)

$$\sum_{i=1}^{k}\sum_{n=2}^{\infty}\{n^k[n(1+\gamma\zeta_{1,}) + \alpha(\gamma\zeta_2 - 1)]\}|C_n^i|$$

$$= \sum_{i=1}^{k}\sum_{n=2}^{\infty}\{n^k[n(1+\gamma\zeta_{1,}) + \alpha(\gamma\zeta_2 - 1)]\}\left|(1-\lambda)\left(a_n^i + \overline{b_n^i}\right) + \lambda\left(A_n^i + \overline{B_n^i}\right)\right|$$

$$= \sum_{i=1}^{k}\sum_{n=2}^{\infty}\{n^k[n(1+\gamma\zeta_{1,}) + \alpha(\gamma\zeta_2 - 1)]\}\left|(1-\lambda)\left(a_n^i + \overline{b_n^i}\right)\right|$$

$$+ \sum_{i=1}^{k}\sum_{n=2}^{\infty}\{n^k[n(1+\gamma\zeta_{1,}) + \alpha(\gamma\zeta_2 - 1)]\}\left|\lambda\left(A_n^i + \overline{B_n^i}\right)\right|$$

$$\le (1-\lambda)2[\gamma|\zeta_1 + \alpha\zeta_2| - |1-\alpha|] + \lambda 2[\gamma|\zeta_1 + \alpha\zeta_2| - |1-\alpha|]$$

$$= 2[\gamma|\zeta_1 + \alpha\zeta_2| - |1-\alpha|]$$

Which proves the theorem.

Theorem 3 Let $D^nF_i(z)$ be as in (6) and $F_\alpha(z)$ be as in (7). Then the function $D^nC_i(z)$

Defined by $D^nC_i(z) = z + \sum_{n=2}^{\infty} n^k a_n^i A_n^i z^n + \overline{\sum_{n=1}^{\infty} n^k b_n^i B_n^i z^n}$ belong to the class $\Gamma_\alpha^n(\zeta_{1,}\zeta_2;\gamma)$ if and only if

$$\sum_{i=1}^{k}\sum_{n=2}^{\infty}\{n^k[n(1+\gamma\zeta_{1,}) + \alpha(\gamma\zeta_2 - 1)]\}|a_n^iA_n^i + b_n^iB_n^i| \le 2[\gamma|\zeta_1 + \alpha\zeta_2| - |1-\alpha|]$$

Proof: The proof follows the procedure of that of the Theorem 1

Corollary 4 Let $D^nF_i(z)$ be as in (6) and $F_\alpha(z)$ be as in (7) and the function $D^nC_i(z)$

Defined by $D^nC_i(z) = z + \sum_{n=2}^{\infty} n^k a_n^i A_n^i z^n + \overline{\sum_{n=1}^{\infty} n^k b_n^i B_n^i z^n}$ belong to the class $\Gamma_\alpha^n(\zeta_{1,}\zeta_2;\gamma)$

Then we have

$$|a_n^iA_n^i + b_n^iB_n^i| \le \frac{2[\gamma|\zeta_1 + \alpha\zeta_2| - |1-\alpha|]}{n^k[n(1+\gamma\zeta_{1,}) + \alpha(\gamma\zeta_2 - 1)]}$$

Corollary 5 Let $D^nF_i(z)$ be as in (6) and $F_\alpha(z)$ be as in (7) and the function $D^nC_i(z)$

Defined by $D^n C_i(z) = z + \sum_{n=2}^{\infty} n^k a_n^i A_n^i z^n + \overline{\sum_{n=1}^{\infty} n^k b_n^i B_n^i z^n}$ belong to the class $\Gamma_\alpha^n(\zeta_{1,}\zeta_2;\gamma)$

Then we have

$$|a_n^i A_n^i| \le \frac{2[\gamma|\zeta_1 + \alpha\zeta_2| - |1-\alpha|] - |b_n^i B_n^i| n^k [n(1+\gamma\zeta_{1,}) + \alpha(\gamma\zeta_2 - 1)]}{n^k [n(1+\gamma\zeta_{1,}) + \alpha(\gamma\zeta_2 - 1)]}$$

Corollary 6 Let $D^n F_i(z)$ be as in (6) and $F_\alpha(z)$ be as in (7) and the function $D^n C_i(z)$

Defined by $D^n C_i(z) = z + \sum_{n=2}^{\infty} n^k a_n^i A_n^i z^n + \overline{\sum_{n=1}^{\infty} n^k b_n^i B_n^i z^n}$ belong to the class $\Gamma_\alpha^n(\zeta_{1,}\zeta_2;\gamma)$

Then we have

$$|b_n^i B_n^i| \le \frac{2[\gamma|\zeta_1 + \alpha\zeta_2| - |1-\alpha|] - |a_n^i A_n^i| n^k [n(1+\gamma\zeta_{1,}) + \alpha(\gamma\zeta_2 - 1)]}{n^k [n(1+\gamma\zeta_{1,}) + \alpha(\gamma\zeta_2 - 1)]}$$

Corollary 7 Let $D^n F_i(z)$ be as in (6) and $F_\alpha(z)$ be as in (7) and the function $D^n C_i(z)$

Defined by $D^n C_i(z) = z + \sum_{n=2}^{\infty} n^k a_n^i A_n^i z^n + \overline{\sum_{n=1}^{\infty} n^k b_n^i B_n^i z^n}$ belong to the class $\Gamma_\alpha^n(\zeta_{1,}\zeta_2;\gamma)$

Then we have

$$|a_n^i| \le \frac{2[\gamma|\zeta_1 + \alpha\zeta_2| - |1-\alpha|] - |b_n^i B_n^i| n^k [n(1+\gamma\zeta_{1,}) + \alpha(\gamma\zeta_2 - 1)]}{|A_n^i| n^k [n(1+\gamma\zeta_{1,}) + \alpha(\gamma\zeta_2 - 1)]}$$

Corollary 8 Let $D^n F_i(z)$ be as in (6) and $F_\alpha(z)$ be as in (7) and the function $D^n C_i(z)$

Defined by $D^n C_i(z) = z + \sum_{n=2}^{\infty} n^k a_n^i A_n^i z^n + \overline{\sum_{n=1}^{\infty} n^k b_n^i B_n^i z^n}$ belong to the class $\Gamma_\alpha^n(\zeta_{1,}\zeta_2;\gamma)$

Then we have

$$|A_n^i| \le \frac{2[\gamma|\zeta_1 + \alpha\zeta_2| - |1-\alpha|] - |b_n^i B_n^i| n^k [n(1+\gamma\zeta_{1,}) + \alpha(\gamma\zeta_2 - 1)]}{|a_n^i| n^k [n(1+\gamma\zeta_{1,}) + \alpha(\gamma\zeta_2 - 1)]}$$

Corollary 9 Let $D^n F_i(z)$ be as in (6) and $F_\alpha(z)$ be as in (7) and the function $D^n C_i(z)$Defined by $D^n C_i(z) = z + \sum_{n=2}^{\infty} n^k a_n^i A_n^i z^n + \overline{\sum_{n=1}^{\infty} n^k b_n^i B_n^i z^n}$ belong to the class $\Gamma_\alpha^n(\zeta_{1,}\zeta_2;\gamma)$

Then we have

$$|b_n^i| \le \frac{2[\gamma|\zeta_1 + \alpha\zeta_2| - |1-\alpha|] - |a_n^i A_n^i| n^k [n(1+\gamma\zeta_{1,}) + \alpha(\gamma\zeta_2 - 1)]}{|B_n^i| n^k [n(1+\gamma\zeta_{1,}) + \alpha(\gamma\zeta_2 - 1)]}$$

Corollary 10 Let $D^n F_i(z)$ be as in (6) and $F_\alpha(z)$ be as in (7) and the function $D^n C_i(z)$

Defined by $D^n C_i(z) = z + \sum_{n=2}^{\infty} n^k a_n^i A_n^i z^n + \overline{\sum_{n=1}^{\infty} n^k b_n^i B_n^i z^n}$ belong to the class $\Gamma_\alpha^n(\zeta_{1,}\zeta_2;\gamma)$

Then we have

$$|B_n^i| \leq \frac{2[\gamma|\zeta_1 + \alpha\zeta_2| - |1-\alpha|] - |a_n^i A_n^i| n^k [n(1+\gamma\zeta_{1,}) + \alpha(\gamma\zeta_2 - 1)]}{|b_n^i| n^k [n(1+\gamma\zeta_{1,}) + \alpha(\gamma\zeta_2 - 1)]}$$

Remarks: It is noted that $\Gamma_1^0(0,1;1)$ is equivalent to the well known classes of starlike function S^* given by

$$S^* = \left\{ f_i : Re\left\{\frac{zf'_i(z)}{f_i(z)}\right\} > 0, z \in U \right\}$$

And that $\Gamma_1^1(0,1;1)$ is equivalent to the well known classes of convex function S^c given b $S^c =$

$$\left\{ f_i : Re\left\{1 + \frac{zf''_i(z)}{f'_i(z)}\right\} > 0, z \in U \right\}$$

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