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Single Server Bulk Queueing System with Three Stage Heterogeneous Service, Compulsory Vacation and Balking

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Abstract- The concept of this paper studies with the customers arriving in bulk or group, in a single server queueing system, in Poisson distribution which provides three types of general services in bulk of fixed size M (≥ 1) in first come first served basis. After first two stage service, the server must provide the third stage service. After the completion of third stage service, the server takes compulsory vacation under exponential distribution. If the required bulk of customers are not available on the return of the server, the server again goes for vacation or remains in the system till bulk is reached. The arriving batch balks during the period when the server is busy or when the server is on vacation or other constraints. This may result in the impatient behavior of the customers. From the above concept, we compute the time dependent probability generating functions and from it the corresponding steady state results are obtained. The average queue size and the system size are derived.

Key words - balking, bulk arrival, bulk service, probability generating function, vacation.

I. Introduction

Among the eminent researchers, Chaudhry and Templeton [4], Cooper [5], Gross and Harris [9] have widely done research in bulk queue. Vacation queue has been surveyed widely by Doshi [7]. Madan [11] has been explained the queueing system with compulsory vacation. Thangaraj and Vanitha [15] have explained M/G/1 queue with two stage heterogeneous service compulsory vacation. Uma and Punniyamoorthy [17] have been worked the bulk queueing system with two choices of service and compulsory vacation. Ayyappan and Sathiya [2] have briefed the three stage heterogeneous service and server vacations. Queueing system with balking and vacation have detailed by Charan Jeet Singh, Madhu Jain and Binay Kumar [3], Monita Baruah, Madan and Tillal Eldabi [14], Punniyamoorthy and Uma [15].

This paper, we suggest to study single server bulk queue with three stages of service and compulsory vacation. The arrival is under Poisson distribution, the services are general distribution and vacation is an exponential distribution. After the second stage service, the server must provide the third stage service. The bulk of customers are served under first come first served basis. The arriving batch balks when the server is busy or when the server is on vacation or other constraints. This may reflect customer's impatient behavior. Queueing systems with impatient customers appear in many real life situations such as those involving production-inventory systems, telephone switching systems, hospital emergency service and so forth.

This paper is organized as follows: The mathematical model is briefed in section 2. Definitions and Notations are described in section 3. Equations governing the system are given in section 4. The time dependent solutions have been discussed in section 5 and the corresponding steady state results have been calculated clearly in section 6. The average queue size and the system size are calculated in section 7.

II. THE MATHEMATICAL MODEL

We assume the following to describe the queueing model of our study:

1) Customers (units) arrive at the system in batches of variable size in a compound Poisson process.

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2) Let $\lambda\pi_i dt$ $(i = 1,2,3,\dots)$ be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t+dt]$, where $0 \le \pi_i \le 1$, $\sum_{i=1}^{\infty}\pi_i = 1$, $\lambda > 0$ is the mean arrival rate of batches.
3) We consider the case when there is single server providing parallel service of three types on a first come first served basis (FCFS); after first two stages of service, the server must provide the third stage service
4) The service of customers (units) is rendered in batches of fixed size $M(\ge 1)$ or $\min(n, M)$, where n is the number of customers in the queue.
5) We assume that the random variable of service time S_j $(j = 1,2)$ of the j^{th} kind of service follows a general probability law with distribution function $G_j(s_j)$, $g_j(s_j)$ is the probability density function and $E(S_j^k)$ is the k^{th} moment $(k = 1,2,\dots)$ of service time $j = 1,2,3$.
6) Let $\mu_j(x)$ be the conditional probability of type j service completion during the period $(x, x+dx]$, given that elapsed service time is x, so that

$$\mu_j(x) = \frac{g_j(x)}{1-G_j(x)}, j = 1,2 \tag{1}$$

and therefore

$$g_j(s_j) = \mu_j(s_j)e^{-\int_0^{s_j}\mu_j(x)dx}, j = 1,2 \tag{2}$$

7) After completion of continuous service to the batches of fixed size $M(\ge 1)$, the server will go for compulsory vacation.
8) Server vacation starts after the completion of service to a batch. The duration of the vacation period is assumed to be exponential with mean vacation time $\frac{1}{\alpha}$.
9) On returning from vacation the server instantly starts the service if there is a batch of size M or he remains idle in the system.
10) We assume that $(1-a_1)(0 \le a_1 \le 1)$ is the probability that an arriving batch balks during the period when the server is busy (available on the system) and $(1-a_2)(0 \le a_2 \le 1)$ is the probability that an arriving batch balks during the period when the server is on vacation.
11) Finally, it is assumed that the inter-arrival times of the customers, the service times of each kind of service and vacation times of the server, all these stochastic processes involved in the system are independent of each other.

III. DEFINITIONS AND NOTATIONS

We define:

$P_{n,j}(x,t)$: Probability that at time t, the server is active providing and there are n $(n \ge 0)$ customers in the queue, excluding a batch of M customers in type j service, $j = 1,2,3$ and the elapsed service time of this customer is x. Accordingly, $P_{n,j}(t) = \int_0^{\infty} P_{n,j}(x,t)dx$ denotes the probability that there are n customers in the queue excluding a batch of M customers in type j service, $j = 1,2,3$ irrespective of the elapsed service time x.

$V_n(t)$: Probability that at time t, there are n $(n \ge 0)$ customers in the queue and the server is on vacation.

$Q(t)$: Probability that at time t, there are less than M customers in the system and the server is idle but available in the system.

IV. EQUATIONS GOVERNING THE SYSTEM

According to the Mathematical model mentioned above, the system has the following set of differential-difference equations:

$$\frac{\partial}{\partial x}P_{n,1}(x,t) + \frac{\partial}{\partial t}P_{n,1}(x,t) + \big(\lambda + \mu_1(x)\big)P_{n,1}(x,t) = \lambda a_1 \sum_{k=1}^{n}\pi_k P_{n-k,1}(x,t) + \lambda(1-a_1)P_{n,1}(x,t) \tag{3}$$

$$\frac{\partial}{\partial x}P_{0,1}(x,t) + \frac{\partial}{\partial t}P_{0,1}(x,t) + \big(\lambda + \mu_1(x)\big)P_{0,1}(x,t) = \lambda(1-a_1)P_{0,1}(x,t) \tag{4}$$

$$\frac{\partial}{\partial x}P_{n,2}(x,t) + \frac{\partial}{\partial t}P_{n,2}(x,t) + \big(\lambda + \mu_2(x)\big)P_{n,2}(x,t) = \lambda a_1 \sum_{k=1}^{n}\pi_k P_{n-k,2}(x,t) + \lambda(1-a_1)P_{n,2}(x,t) \tag{5}$$

$$\frac{\partial}{\partial x}P_{0,2}(x,t) + \frac{\partial}{\partial t}P_{0,2}(x,t) + \big(\lambda + \mu_2(x)\big)P_{0,2}(x,t) = \lambda(1-a_1)P_{0,2}(x,t) \tag{6}$$

$$\frac{\partial}{\partial x}P_{n,3}(x,t) + \frac{\partial}{\partial t}P_{n,3}(x,t) + \big(\lambda + \mu_3(x)\big)P_{n,3}(x,t) = \lambda a_1 \sum_{k=1}^{n}\pi_k P_{n-k,3}(x,t) + \lambda(1-a_1)P_{n,3}(x,t) \tag{7}$$

$$\frac{\partial}{\partial x}P_{0,3}(x,t) + \frac{\partial}{\partial t}P_{0,3}(x,t) + \big(\lambda + \mu_3(x)\big)P_{0,3}(x,t) = \lambda(1-a_1)P_{0,3}(x,t) \tag{8}$$

$$\frac{d}{dt}V_n(t) + (\lambda+\alpha)V_n(t) = \lambda a_2 \sum_{k=1}^{n}\pi_k V_{n-k}(t) + \lambda(1-a_2)V_n(t) + \int_0^{\infty} P_{n,3}(x,t)\mu_3(x)dx \tag{9}$$

$$\frac{d}{dt}V_0(t) + (\lambda+\alpha)V_0(t) = \lambda(1-a_2)V_0(t) + \int_0^{\infty} P_{0,3}(x,t)\mu_3(x)dx \tag{10}$$

$\frac{d}{dt}Q(t) + \lambda Q(t) = \alpha V_0(t) + \lambda(1 - a_1)Q(t)$ (11)

Equations (5) – (11) are to be solved subject to the following boundary conditions:

$P_{n,1}(0,t) = \alpha V_{n+M}(t)$ (12)

$P_{0,1}(0,t) = \alpha \sum_{b=1}^{M} V_b(t) + \lambda a_1 Q(t)$ (13)

$P_{n,2}(0,t) = \int_0^\infty P_{n,1}(x,t)\mu_1(x)dx$ (14)

$P_{n,3}(0,t) = \int_0^\infty P_{n,2}(x,t)\mu_2(x)dx$ (15)

We assume that initially the server is available but idle because of less than M customers so that the initial conditions are

$V_n(0) = 0;\ V_0(0) = 0;\ Q(0) = 1$

$P_{n,j}(0) = 0$, for $n = 0,1,2, \dots$ and $j = 1,2,3$. (16)

V. PROBABILITY GENERATING FUNCTION OF THE QUEUE SIZE:THE TIME DEPENDENT SOLUTION

We define the following probability generating functions:

$$\left.\begin{aligned} P_j(x,z,t) &= \sum_{n=0}^{\infty} P_{n,j}(x,t)z^n, \quad j = 1,2,3 \\ P_j(z,t) &= \sum_{n=0}^{\infty} P_{n,j}(t)z^n, \quad j = 1,2,3 \\ V(z,t) &= \sum_{n=0}^{\infty} V_n(t)z^n \\ \pi(z) &= \sum_{n=1}^{\infty} \pi_n z^n \end{aligned}\right\} \quad (17)$$

Define the Laplace-Stieltjes Transform of a function $f(t)$ as follows:

$\bar{f}(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$ (18)

Taking Laplace Transform of equations (3) – (15) and using (16), we get,

$\frac{\partial}{\partial x}\bar{P}_{n,1}(x,s) + (s + \lambda + \mu_1(x))\bar{P}_{n,1}(x,s) = \lambda a_1 \sum_{k=1}^{n} \pi_k \bar{P}_{n-k,1}(x,s) + \lambda(1 - a_1)\bar{P}_{n,1}(x,s)$ (19)

$\frac{\partial}{\partial x}\bar{P}_{0,1}(x,s) + (s + \lambda + \mu_1(x))\bar{P}_{0,1}(x,s) = \lambda(1 - a_1)\bar{P}_{0,1}(x,s)$ (20)

$\frac{\partial}{\partial x}\bar{P}_{n,2}(x,s) + (s + \lambda + \mu_2(x))\bar{P}_{n,2}(x,s) = \lambda a_1 \sum_{k=1}^{n} \pi_k \bar{P}_{n-k,2}(x,s) + \lambda(1 - a_1)\bar{P}_{n,2}(x,s)$ (21)

$\frac{\partial}{\partial x}\bar{P}_{0,2}(x,s) + (s + \lambda + \mu_2(x))\bar{P}_{0,2}(x,s) = \lambda(1 - a_1)\bar{P}_{0,2}(x,s)$ (22)

$\frac{\partial}{\partial x}\bar{P}_{n,3}(x,s) + (s + \lambda + \mu_3(x))\bar{P}_{n,3}(x,s) = \lambda a_1 \sum_{k=1}^{n} \pi_k \bar{P}_{n-k,3}(x,s) + \lambda(1 - a_1)\bar{P}_{n,3}(x,s)$ (23)

$\frac{\partial}{\partial x}\bar{P}_{0,3}(x,s) + (s + \lambda + \mu_3(x))\bar{P}_{0,3}(x,s) = \lambda(1 - a_1)\bar{P}_{0,3}(x,s)$ (24)

$(s + \lambda + \alpha)\bar{V}_n(s) = \lambda a_2 \sum_{k=1}^{n} \pi_k \bar{V}_{n-k}(s) + \lambda(1 - a_2)\bar{V}_n(s) + \int_0^\infty \bar{P}_{n,3}(x,s)\mu_3(x)dx$ (25)

$(s + \lambda + \alpha)\bar{V}_0(s) = \lambda(1 - a_2)\bar{V}_0(s) + \int_0^\infty \bar{P}_{0,3}(x,s)\mu_3(x)dx$ (26)

$(s + \lambda)\bar{Q}(s) = 1 + \alpha\bar{V}_0(s) + \lambda(1 - a_1)\bar{Q}(s)$ (27)

$\bar{P}_{n,1}(0,s) = \alpha\bar{V}_{n+M}(s)$ (28)

$\bar{P}_{0,1}(0,s) = \alpha \sum_{b=1}^{M} \bar{V}_b(s) + \lambda a_1 \bar{Q}(s)$ (29)

$\bar{P}_{n,2}(0,s) = \int_0^\infty \bar{P}_{n,1}(x,s)\,\mu_1(x)dx$ (30)

$\bar{P}_{n,3}(0,s) = \int_0^\infty \bar{P}_{n,2}(x,s)\,\mu_2(x)dx$ (31)

Multiplying the equation (19) by z^n and summing over n from 1 to ∞, adding equation (20) and using the generating functions defined in (17), we obtain,

$\frac{\partial}{\partial x}\bar{P}_1(x,z,s) + \{s + \lambda a_1(1 - \pi(z)) + \mu_1(x)\}\bar{P}_1(x,z,s) = 0$ (32)

Performing similar operations on equations (21) – (26), we obtain,

$\frac{\partial}{\partial x}\bar{P}_2(x,z,s) + \{s + \lambda a_1(1 - \pi(z)) + \mu_2(x)\}\bar{P}_2(x,z,s) = 0$ (33)

$\frac{\partial}{\partial x}\bar{P}_3(x,z,s)+\{s+\lambda a_1(1-\pi(z))+\mu_3(x)\}\bar{P}_3(x,z,s)=0$ (34)

$\{s+\lambda a_2(1-\pi(z))+\alpha\}\bar{V}(z,s)=\int_0^\infty \bar{P}_3(x,z,s)\mu_3(x)dx$ (35)

Multiplying the equation (28) by z^{n+M} and summing over n from 1 to ∞ and adding, multiplying the equation (29) by z^M, and using the generating functions defined in (17) and using (27), we obtain,

$\bar{P}_1(0,z,s)=z^{-M}\alpha\,\bar{V}(z,s)+\alpha\sum_{b=1}^{M-1}(1-z^{-M+b})\bar{V}_b(s)+[\lambda a_1-(s+\lambda)a_1z^{-M}]\bar{Q}(s)+z^{-M}$ (36)

Multiplying the equation (30) by z^n and summing over n from 0 to ∞ and using the generating functions defined in (17), we obtain,

$\bar{P}_2(0,z,s)=\int_0^\infty \bar{P}_1(x,z,s)\mu_1(x)dx$ (37)

Performing similar operations on equation (31), we obtain,

$\bar{P}_3(0,z,s)=\int_0^\infty \bar{P}_2(x,z,s)\mu_2(x)dx$ (38)

We now integrate equations (32) - (34) between the limits 0 and x and obtain,

$\bar{P}_1(x,z,s)=\bar{P}_1(0,z,s)e^{-Rx-\int_0^x \mu_1(x)dx}$ (39)

$\bar{P}_2(x,z,s)=\bar{P}_2(0,z,s)e^{-Rx-\int_0^x \mu_2(x)dx}$ (40)

$\bar{P}_3(x,z,s)=\bar{P}_3(0,z,s)e^{-Rx-\int_0^x \mu_3(x)dx}$ (41)

Where $R=s+\lambda a_1(1-\pi(z))$

Integrating equations (39) - (41) by parts, with respect to x, we get,

$\bar{P}_1(z,s)=\bar{P}_1(0,z,s)\left[\frac{1-\bar{G}_1(R)}{R}\right]$ (42)

$\bar{P}_2(z,s)=\bar{P}_2(0,z,s)\left[\frac{1-\bar{G}_2(R)}{R}\right]$ (43)

$\bar{P}_3(z,s)=\bar{P}_3(0,z,s)\left[\frac{1-\bar{G}_3(R)}{R}\right]$ (44)

Where $\bar{G}_j(R)=\int_0^\infty e^{-R\,x}dG_j(x)$, is the Laplace Transform of j^{th} type of service, $j=1,2,3$.

Multiplying the equations (39), (40) and (41) by $\mu_1(x)$, $\mu_2(x)$and $\alpha(x)$ integrating by parts, with respect to x, we get,

$\int_0^\infty \bar{P}_1(x,z,s)\mu_1(x)dx=\bar{P}_1(0,z,s)\bar{G}_1(R)$ (45)

$\int_0^\infty \bar{P}_2(x,z,s)\mu_2(x)dx=\bar{P}_2(0,z,s)\bar{G}_2(R)$ (46)

$\int_0^\infty \bar{P}_3(x,z,s)\mu_3(x)dx=\bar{P}_3(0,z,s)\bar{G}_3(R)$ (47)

Substituting (45) in (37) we get,

$\bar{P}_2(0,z,s)=\bar{P}_1(0,z,s)\bar{G}_1(R)$ (48)

Substituting (36) in (48) we get,

$\bar{P}_2(0,z,s)=[z^{-M}\alpha\,\bar{V}(z,s)+\alpha\sum_{b=1}^{M-1}(1-z^{-M+b})\bar{V}_b(s)+[\lambda a_1-(s+\lambda)a_1z^{-M}]\bar{Q}(s)+z^{-M}]\bar{G}_1(R)$ (49)

Substituting (46) in (38) we get,

$\bar{P}_3(0,z,s)=\bar{P}_2(0,z,s)\bar{G}_2(R)$ (50)

Substituting (49) in (50) we get,

$\bar{P}_3(0,z,s)=[z^{-M}\alpha\,\bar{V}(z,s)+\alpha\sum_{b=1}^{M-1}(1-z^{-M+b})\bar{V}_b(s)+[\lambda a_1-(s+\lambda)a_1z^{-M}]\bar{Q}(s)+z^{-M}]\bar{G}_1(R)\bar{G}_2(R)$ (51)

Substituting (36), (49) and (50) in (42), (43) and (44) respectively, we get,

$\bar{P}_1(z,s)=[z^{-M}\alpha\,\bar{V}(z,s)+\alpha\sum_{b=1}^{M-1}(1-z^{-M+b})\bar{V}_b(s)+[\lambda a_1-(s+\lambda)a_1z^{-M}]\bar{Q}(s)+z^{-M}]\left[\frac{1-\bar{G}_1(R)}{R}\right]$ (52)

$\bar{P}_2(z,s)=[\{z^{-M}\alpha\,\bar{V}(z,s)+\alpha\sum_{b=1}^{M-1}(1-z^{-M+b})\bar{V}_b(s)+[\lambda a_1-(s+\lambda)a_1z^{-M}]\bar{Q}(s)+z^{-M}\}\bar{G}_1(R)]\left[\frac{1-\bar{G}_2(R)}{R}\right]$ (53)

$\bar{P}_3(z,s)=[\{z^{-M}\alpha\,\bar{V}(z,s)+\alpha\sum_{b=1}^{M-1}(1-z^{-M+b})\bar{V}_b(s)+[\lambda a_1-(s+\lambda)a_1z^{-M}]\bar{Q}(s)+z^{-M}\}\bar{G}_1(R)\bar{G}_2(R)]\left[\frac{1-\bar{G}_3(R)}{R}\right]$(54)

Substituting (47) in (35) and using (51), we get,

$$\bar{V}(z,s) = \frac{\{\alpha\sum_{b=1}^{M-1}(1-z^{-M+b})\bar{V}_b(s)+[\lambda-(s+\lambda)z^{-M}]\bar{Q}(s)+z^{-M}\}\bar{G}_1(R)\bar{G}_2(R)\bar{G}_3(R)}{\{s+\lambda a_2(1-\pi(z))\}+\alpha-z^{-M}\alpha\bar{G}_1(R)\bar{G}_2(R)\bar{G}_3(R)} \quad (55)$$

We note that there are M unknowns, $\bar{Q}(s)$and $\bar{V}_b(s), b = 1,2,\dots,M-1$ appearing in equation (55).

Now, (55) gives the probability generating function of the service system with M unknowns. By Rouche's theorem of complex variables, it can be proved that $\{s+\lambda a_2(1-\pi(z))\}+\alpha-z^{-M}\alpha\bar{G}_1(R)\bar{G}_2(R)\bar{G}_3(R)$ has M zeroes inside the contour $|z| = 1$. Since $\bar{P}_1(z,s),,\bar{P}_2(z,s),\bar{P}_3(z,s)$ and $\bar{V}(z,s)$ are analytic inside the unit circle $|z| = 1$, the numerator in the right hand side of equations(55) must vanish at these points, which gives rise to a set of M linear equations which are sufficient to determine Munknowns.

VI. THE STEADY STATE RESULTS

To define the steady state probabilities and corresponding generating functions, we drop the argument t, and for that matter the argument s wherever it appears in the time-dependent analysis up to this point. Then the corresponding steady state results can be obtained by using the well-known Tauberian Property

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t) \quad (56)$$

if the limit on the right exists.

Now (52), (53), (54) and (55) we have,

$$P_1(z) = \{\alpha z^{-M}V(z)+\alpha z^{-M}U+\lambda a_1(1-z^{-M})Q\}\left\{\frac{1-\bar{G}_1(f(z))}{f(z)}\right\} \quad (57)$$

$$P_2(z) = \{\alpha z^{-M}V(z)+\alpha z^{-M}U+\lambda a_1(1-z^{-M})Q\}\bar{G}_1(f(z))\left\{\frac{1-\bar{G}_2(f(z))}{f(z)}\right\} \quad (58)$$

$$P_3(z) = \{\alpha z^{-M}V(z)+\alpha z^{-M}U+\lambda a_1(1-z^{-M})Q\}\bar{G}_1(f(z))\bar{G}_2(f(z))\left\{\frac{1-\bar{G}_3(f(z))}{f(z)}\right\} \quad (59)$$

$$V(z) = \frac{\{\alpha U+\lambda(z^M-1)Q\}\bar{G}_1(f(z))\bar{G}_2(f(z))\bar{G}_3(f(z))}{z^M f_1(z)+z^M\alpha-\alpha\bar{G}_1(f(z))\bar{G}_2(f(z))\bar{G}_3(f(z))} \quad (60)$$

The M unknowns, Q and $V_b, b = 1,2,\dots,M-1$can be determined as before.
Where $f(z) = \lambda a_1(1-\pi(z))$; $f_1(z) = \lambda a_2(1-\pi(z))$ and $U = \sum_{b=1}^{M-1}(z^M-z^b)V_b$

Let $A_q(z)$ denote the probability generating function of the queue size irrespective of the state of the system.

$$\text{i.e.}, A_q(z) = P_1(z)+P_2(z)+P_3(z)+V(z) \quad (61)$$

In order to find Q, we use the normalization condition

$$A_q(1)+Q = 1 \quad (62)$$

Note that for $z = 1$, $A_q(1)$ is indeterminate of $\frac{0}{0}$ form.
Therefore, we apply L'Hôpital's Rule on (61), we get,

$$A_q(1) = \frac{E^*[\alpha U+\lambda a_1 MQ]}{M\alpha-\lambda E(I)\{a_2+\alpha a_1 E_1\}} \quad (63)$$

Where $E^* = \alpha E(S_1)+\alpha E(S_2)+\alpha E(S_3)+1$ and $E_1 = E(S_1)+E(S_2)+E(S_3)$
We used $\bar{G}_j(0) = 1,\ j = 1,2,3, \bar{V}(0) = 1, \pi'(1) = E(I)$, where I denotes the number of customers in an arriving batch and therefore, $E(I)$ is the mean of the batch size of the arriving customers. Similarly $E(S_1),\ E(S_2), E(S_3)$ are the mean service times of type 1, type 2, type 3 services, respectively.

Therefore, adding Q to equation (63) and equating to 1 and simplifying we get,

$$Q = 1-\frac{E^*(M\lambda+\alpha U)}{M\alpha+\lambda a_1 ME^*-\lambda E(I)[\alpha a_1 E_1+a_2]} \quad (64)$$

Equation (64) gives the probability that the server is idle.

From equation (64) the utilization factor, ρ of the system is given by

$$\rho = \frac{E^*(M\lambda+\alpha U)}{M\alpha+\lambda a_1 ME^*-\lambda E(I)[\alpha a_1 E_1+a_2]} \quad (65)$$

Where $\rho < 1$ is the stability condition under which the steady state exists.

VII. THE AVERAGE QUEUE SIZE AND THE SYSTEM SIZE

Let L_q denote the mean number of customers in the queue under the steady state.

Then $L_q = \frac{d}{dz} A_q(z)\Big|_{z=1}$ (66)

Since the formula gives indeterminate form, then we write $A_q(z)$ as

$$A_q(z) = \frac{N(z)}{D(z)} + C(z)$$

Where $N(z)$ and $D(z)$ are the numerator and denominator of the first term and $C(z)$ is the second term of the right hand side of $A_q(z)$ respectively.

Then using L'Hôpital's Rule twice we obtain,

$$L_q = \lim_{z\to 1} \frac{D'(z)N''(z)-N'(z)D''(z)}{2\left(D'(z)\right)^2} + \frac{d}{dz}C(z)|_{z=1} \quad (67)$$

Where primes and double primes in (67) denote first and second derivatives at $z = 1$, respectively. Carrying out the derivatives at $z = 1$ we have,

$$N'(1) = E^*[\alpha U + \lambda a_1 MQ] \quad (68)$$

$$D'(1) = \alpha M - \lambda E(I)(a_2 + \alpha a_1 E_1) \quad (69)$$

$$N''(1) = [2\lambda a_1 E(I)E^* E_1 + \alpha\{\lambda a_1 E(I)(E^{**} + 2E_1^*) - 2ME_1\}][\alpha U + \lambda a_1 MQ] + U^* E^* \quad (70)$$

$$D''(1) = \alpha M(M-1) - 2\lambda a_2 ME(I) - \lambda E\big(I(I-1)\big)\{a_2 + \alpha a_1 E_1\} - \alpha\lambda^2 a_1^2 \big(E(I)\big)^2\{E^{**} + 2E_1^*\} \quad (71)$$

$$\frac{d}{dz}C(z)|_{z=1} = E_1(Ma_1\lambda Q + \alpha U) \quad (72)$$

Therefore, the mean number of customers in the queue is

$$L_q = \frac{\{\alpha M-\lambda E(I)(a_2+\alpha a_1 E_1)\}\left\{\left[\begin{matrix} 2\lambda a_1 E(I)E^* E_1 \\ +\alpha\left\{\begin{matrix}\lambda a_1 E(I)(E^{**}+2E_1^*) \\ -2ME_1\end{matrix}\right\}\end{matrix}\right][\alpha U+\lambda a_1 MQ]+U^* E^*\right\} - E^*[\alpha U+\lambda a_1 MQ]\left\{\begin{matrix}\alpha M(M-1)-2\lambda a_2 ME(I) \\ -\lambda E\big(I(I-1)\big)\{a_2+\alpha a_1 E_1\} \\ -\alpha\lambda^2 a_1^2\big(E(I)\big)^2\{E^{**}+2E_1^*\}\end{matrix}\right\}}{2\{\alpha M-\lambda E(I)(a_2+\alpha a_1 E_1)\}^2} \quad (73)$$

Where $E_1^* = E(S_1)E(S_2) + E(S_2)E(S_3) + E(S_3)E(S_1)$,
$E^{**} = E(S_1^2) + E(S_2^2) + E(S_3^2)$ and $U^* = \alpha\sum_{b=1}^{M-1}\{M(M-1) - b(b-1)\}V_b + \lambda a_1 M(M-1)\,Q$

Also, where $E\big(I(I-1)\big)$ is the second factorial moment of the batch size of the arriving customers. Similarly, $E(S_1^2)$, $E(S_2^2)$ and $E(S_3^2)$ are the second moments of the service times of type 1, type 2 and type 3 services, respectively. Q has been obtained in (64).
Further, the average number of customers in the system can be found as $L_s = L_q + \rho$ by using Little's formula.

VIII. CONCLUSION

The single server bulk queue with three stage heterogeneous, compulsory vacation and balking is discussed. The transient solution, steady state results, the average number of customers in the queue and the system are computed in this paper.

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