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New Subclasses of Meromorphically Multivalent Functions Defined by a Differential Operator

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Abstract-In this paper authors introduced two new subclasses $\sum_{\varrho\tau\varsigma\xi\, mp}(\alpha,\beta,\eta)$and $\sum^{+}_{\varrho\tau\varsigma\xi\, mp}(\alpha,\beta,\eta)$of meromorphically multivalent functions which aredefined by means of a new differential operator. By making use of the principle of differential subordination, authors investigate several inclusion relationships and properties of certain subclasses which are defined here by means of a differential operator. Some results connectedto subordination properties, coefficient estimates, convolutionproperties, integral representation, distortion theorems are obtained.We also extend the familiarconcept of (n,δ) $-$neighborhoods of analyticfunctions to these subclasses of meromorphicallymultivalentfunctions.

Index Terms-Analytic functions, meromorphic functions,multivalent functions, differential operator, subordination, neighborhoods.

I. Introduction

Let $\tilde{A}$ be the class of analytic functions in the unit disk $U = \{z \in C: |z| < 1\}$.Consider$\Omega = nw \in: w(0) = 0$ and $|w(z)| < 1, z \in U$(1.1)the class of Schwarz functions.For $0 \leq \alpha < 1$let$P(\alpha) = p \in \tilde{A} : p(0) = 1$ and $Rep(z) > \alpha, z \in U$.(1.2)
Note that$P = P(0)$ is the well-known Caratheodory class of functions. The classes of Schwarz and Caratheodory functions play an extremely important role in the theory of analytic functions and havebeen studied by many authors.It is easy to see that$p \in P(\alpha)$if and only if$\frac{p(z)-\propto}{1-\propto} \in P$.(1.3)Using the properties of functions in the class P and the above condition, the following properties of the functions in the class *P(α)*can be obtained.

Lemma 1.1 Let $p \in \tilde{A}$. Then$p \in P(\alpha)$ if and only if there exists$w \in \Omega$such that$p(z) = \frac{1-(4(1-\eta)\propto-1)w(z)}{1-w(z)}$ (1.4)

Lemma 1.2 (Herglotz formula) A function$p \in \tilde{A}$belongs to the class$P(\alpha)$if and only if there exists a probability measure$\mu(x)$ on ∂Usuchthat$p(z) = \int_{|x|} \frac{1-(4(1-\eta)\propto-1)x(z)}{1-x(z)} d\mu(x)$ $(z \in U)$. (1.5)The correspondence between $P(\alpha)$and probability measure $\mu(x)$ on ∂U, given by (1.5) is one-to-one.If$f$and$g$arein$\tilde{A}$ .We say that f is subordinate to$g$, written $f \in g$,if there exists a function $w \in \Omega$such that $f(z) = g(w(z))$$(z \in U)$.It is known that if $f \in g$, then$f(0) = g(0)$ and $f(U) \subset g(U)$. Inparticular, if $g$is univalent in U we have the following equivalence:$f(z) \prec g(z)$ *(z ∈U)* if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.Let$\sum_p$denote the class of all meromorphic functions f of the form$f(z) = z^{-p} + \sum_{k=1-p}^{\infty} a_k z^k$ $(a_k \geq 0,\ p \in N)$ (1.6)which are analytic and p-valent in the punctured unit disk$U^{*} = U\backslash\{0\}$.Denote by $\sum_p^{+}$ the subclass of $\sum_p$consisting of functions of the form$f(z) = z^{-p} + \sum_{k=1-p}^{\infty} a_k z^k \geq 0$ $(z \in U^{*})$(1.7)A function $f \in \sum_p$ is meromorphically multivalent starlike of order*α (0 ≤ α < 1)* (see [2]) if$-Re\left\{\frac{1}{p}\frac{zf'(z)}{f(z)}\right\} > \propto$ $(z \in U)$.

The class of all such functions is denoted by$\sum_p^{+}(\propto)$.If$f \in \sum_p$ is given by (1.6) and $g \in \sum_p$is given by

$$f(z) = z^{-p} + \sum_{k=1-p}^{\infty} b_k z^k$$

then the Hadamard product (or convolution) of $f$and$g$ is defined*by*$(f * g)(z) = z^{-p} + \sum_{k=1-p}^{\infty} a_k b_k z^k \geq 0$ $(p \in$ *N, z∈U*.* For a function *f∈Σp*, we define the differential operator $D^m_{\varrho\tau\varsigma\ \xi p}$ inthe following way:$D^0_{\varrho\tau\varsigma\ \xi p} f(z) = f(z)$

$D^1_{\varrho\tau\varsigma\ \xi p} f(z) = D_{\varrho\tau\varsigma\ \xi p} f(z) = (\tau-\xi)(\tau-$ *ςzp+1fz′′zp−1*

$+\frac{(\tau-\xi)-\varrho^2(\tau-\varsigma\ )}{\varrho}\frac{\left[z^{p+1}f(z)\right]'}{z^p} + \left[\frac{\varrho-(\tau-\xi)-\varrho^2(\tau-\varsigma\ )}{\varrho}\right] f(z)$.(1.8) And in general
$D^m_{\varrho\tau\varsigma\xi p} f(z) =$
$D_{\varrho\tau\varsigma\xi p}\left[D^{m-1}_{\varrho\tau\varsigma\xi p} f(z)\right]$,(1.9)
where, $\left(\begin{matrix} 0 < \varrho \leq \frac{1}{2}, 0 \leq \xi < 1, \tau \geq 1, 0 \leq \varsigma < 1, 0 \leq \eta < 1 \\ 0 \leq \propto < 1, 0 < \beta \leq 1 \text{ and} m \in N \end{matrix}\right)$.
If the function$f \in \Sigma_p$is given by (1.6) then, from (1.8) and (1.9),we obtain$D^m_{\varrho\tau\varsigma\xi p} f(z) = z^{-p} + \sum_{k=1-p}^{\infty} \Phi_k(\varrho,\tau,\varsigma,\xi,m,p) a_k z^k(1.10)(m \in N, p \in N, z \in$ *U**,where*Φkϱ,τ,ς,ξ,m,p=*

$\left\{1 + \left[(k+p)\left(\begin{matrix}\frac{(\tau-\xi)-\varrho^2(\tau-\varsigma\ )}{\varrho} \\ +(k+p+1)(\tau-\xi)(\tau-\varsigma)\end{matrix}\right)\right]\right\}^m$.(1.11)From (1.10) it follows that$D^m_{\varrho\tau\varsigma\xi p} f(z)$ can be written interms ofconvolution as$D^m_{\varrho\tau\varsigma\xi p}\ f(z) = (f * h)(z)$(1.12)where$h(z) = z^{-p} + \sum_{k=1-p}^{\infty} \Phi_k(\varrho, \tau, \varsigma, \xi, m, p)z^k$.(1.13)Note that, the case$\xi = \frac{1}{2}$ and $\tau = \varsigma$ of the differential operator$D^m_{\varrho\tau\varsigma\xi\ ,p}$ was introduced by Srivastava and Patel [18]. Making use of the differential operator$D^m_{\varrho\tau\varsigma\xi p}\ f(z)$ for $p = 1$ was considered in [16].For differential operator$D^m_{\varrho\tau\varsigma\xi p}\ f(z)$, we define a new subclassof the function class $\sum_p$as follows.

Definition 1.1 A function $f \in \sum_p$is said to be in the class$\sum_{\varrho\tau\varsigma\xi\ mp}(\alpha, \beta, \eta)$ if it satisfies thecondition$\left|\frac{\frac{z\left[D^m_{\varrho\tau\varsigma\xi p}\ f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p}\ f(z)}+2p(1-\eta)}{\frac{z\left[D^m_{\varrho\tau\varsigma\xi p}\ f(z)\right]}{D^m_{\varrho\tau\varsigma\xi p}\ f(z)}+8p(1-\eta)^2\alpha-2p(1-\eta)}\right| < \beta$for some z ∈ U*.(1.14)

And$\begin{pmatrix} 0 < \varrho \le \frac{1}{2}, 0 \le \xi < 1, \tau \ge 1, 0 \le \varsigma \le 1, \\ 0 \le \eta < 1, 0 \le \propto < 1, 0 < \beta \le 1 \text{and} m \in N \end{pmatrix}$.Note that a special case of the class$\sum_{\varrho\tau\varsigma\xi mp}(\propto, \beta, \eta)$ for $p = 1$ and $m = 0$is the class of meromorphically starlike functions of order $\propto$ and typeβintroduced earlier by Mogra et al. [12]. It is easy to check that for$m = 0$ and $\beta = 1$, the class$\sum_{\varrho\tau\varsigma\xi mp}(\propto, \beta, \eta)$reduces to the class $p*\propto$.We consider another subclass of$\sum_p$given by$\sum^+_{\varrho\tau\varsigma\xi mp}(\propto, \beta, \eta) = \sum^+_{\varrho\tau\varsigma\xi mp}(\propto, \beta) \cap \sum_{\varrho\tau\varsigma\xi mp}(\propto, \beta, \eta)$.(1.15)The main object of this paper is to present a systematic investigationof the classes$\sum_{\varrho\tau\varsigma\xi mp}(\propto, \beta, \eta)and\sum^*_p(\propto, \beta, \eta)$.

II. MAIN RESULTS

2Properties of the class $\sum_{\varrho\tau\varsigma\xi mp}(\propto, \beta, \eta)$We begin this section with a necessary and sufficient condition, interms of subordination, for a function to be in the class $\sum_{\varrho\tau\varsigma\xi mp}(\propto, \beta, \eta)$.

Theorem 2.1: A function$f \in \sum_p$ is in the class$\sum_{\varrho\tau\varsigma\xi mp}(\propto, \beta, \eta)$ if andonly if$\frac{z\left[D^m_{\varrho\tau\varsigma\xi p}\ f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p}\ f(z)} \prec \frac{p\langle 4(1-\eta)\propto-1\rangle\beta z - p}{1-\beta z}$ $(z \in U)$.(2.1)

Proof: Let$f \in \Sigma_{\varrho\tau\varsigma\xi mp}(\propto, \beta, \eta)$. Then, from (1.6), we have$\left|\frac{-\frac{z\left[D^m_{\varrho\tau\varsigma\xi p}\ f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p}\ f(z)}+2p(\eta-1)}{-\frac{z\left[D^m_{\varrho\tau\varsigma\xi p}\ f(z)\right]}{D^m_{\varrho\tau\varsigma\xi p}\ f(z)}+4p(1-\eta)+2p(\eta-1)\langle 4(1-\eta)\propto-1\rangle}\right|^2 < \beta^2$or

$\frac{(1-\beta^2)}{(-\langle 4(1-\eta)\propto-1\rangle)^2-1}\left|-\frac{z\left[D^m_{\varrho\tau\varsigma\xi p}\ f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p}\ f(z)} + 2p(\eta-1)\right|^2 - \frac{2[1+\beta^2(-\langle 4(1-\eta)\propto-1\rangle)]}{(-\langle 4(1-\eta)\propto-1\rangle)^2-1} Re\left\{-\frac{1}{2p(1-\eta)} \cdot \frac{z\left[D^m_{\varrho\tau\varsigma\xi p}\ f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p}\ f(z)}\right\} < \beta^2$Ifβ ≠ 1,

wehave$\left|-\frac{z\left[D^m_{\varrho\tau\varsigma\xi p}\ f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p}\ f(z)} - 2p(1-\eta)\right|^2 - 2\frac{1+\beta^2\langle 4(\eta-1)\propto+1\rangle}{1-\beta^2} Re\left\{-\frac{1}{2p(1-\eta)} \cdot \frac{z\left[D^m_{\varrho\tau\varsigma\xi p}\ f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p}\ f(z)}\right\} + \left[\frac{1+\beta^2\langle 4(\eta-1)\propto+1\rangle)}{1-\beta^2}\right]^2 < \frac{-1-\beta^2\langle 4(\eta-1)\propto+1\rangle}{1-\beta^2} + \left[\frac{1+\beta^2\langle 4(\eta-1)\propto+1\rangle}{1-\beta^2}\right]^2$ That is$\frac{\left|-\frac{z\left[D^m_{\varrho\tau\varsigma\xi p}\ f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p}\ f(z)}+2p(\eta-1)\frac{1+\beta^2\langle 4(\eta-1)\propto+1\rangle}{1-\beta^2}\right|}{\frac{\beta[1+\langle 4(\eta-1)\propto+1\rangle]}{1-\beta^2}} < 1$ (2.2)The above inequality shows that the values region of $F(z) = -\frac{1}{2p(1-\eta)} \cdot \frac{z\left[D^m_{\varrho\tau\varsigma\xi p}\ f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p}\ f(z)}$is contained in the disk centered at$\frac{1+\beta^2\langle 4(\eta-1)\propto+1\rangle}{1-\beta^2}$and radius $\frac{\beta[1+\langle 4(\eta-1)\propto+1\rangle]}{1-\beta^2}$.Itis easyto check that the function$G(z) = \frac{1-\langle 4(1-\eta)\propto-1\rangle\beta z}{1-\beta z}$ maps the unitdiskU onto the disk $\frac{\left|w-\frac{1+\beta^2\langle 4(\eta-1)\propto+1\rangle}{1-\beta^2}\right|}{\frac{\beta[1+\langle 4(\eta-1)\propto+1\rangle]}{1-\beta^2}} < 1$. Since G is univalent and $F(0) = G(0)$, $F(U) \subset G(U)$,

we obtain that$F(z) \prec G(z)$,that is$-\frac{z\left[D^m_{\varrho\tau\varsigma\xi p}\ f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p}\ f(z)} \prec 2p(1-\eta)\frac{1-\langle 4(1-\eta)\propto-1\rangle\beta z}{1-\beta z}$ or $\frac{z\left[D^m_{\varrho\tau\varsigma\xi p}\ f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p}\ f(z)} \prec \frac{-2p(1-\eta)+2p(1-\eta)\langle 4(1-\eta)\propto-1\rangle\beta z}{1-\beta z\beta z}$.Conversely, suppose that subordination$\frac{z\left[D^m_{\varrho\tau\varsigma\xi p}\ f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p}\ f(z)} \prec \frac{p\langle 4(1-\eta)\propto-1\rangle\beta z - p}{1-\beta z}$ holds. Then

$-\frac{z\left[D^m_{\varrho\tau\varsigma\xi p}\ f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p}\ f(z)} = 2p(1-\eta)\frac{1-\langle 4(1-\eta)\propto-1\rangle\beta w(z)\beta z}{1-\beta w(z)\beta z}$,(2.3)

where$w \in \Omega$. After simple calculations, from (2.3), we obtain$\left|\frac{\frac{z\left[D^m_{\varrho\tau\varsigma\xi p}\ f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p}\ f(z)}+2p(1-\eta)}{\frac{z\left[D^m_{\varrho\tau\varsigma\xi p}\ f(z)\right]}{D^m_{\varrho\varsigma\tau\xi p}\ f(z)}+2p(1-\eta)\langle 4(1-\eta)\propto-1\rangle}\right| < \beta$which proves that $f \in \sum_{\varrho\tau\varsigma\xi mp}(\propto, \beta, \eta)$.If $\beta = 1$, inequality (1.14) becomes$\left|\frac{-\frac{z\left[D^m_{\varrho\tau\varsigma\xi p}\ f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p}\ f(z)}+2p(\eta-1)}{-\frac{z\left[D^m_{\varrho\tau\varsigma\xi p}\ f(z)\right]}{D^m_{\varrho\tau\varsigma\xi p}\ f(z)}+2p(1-\eta)\langle 4(1-\eta)\propto-1\rangle}\right| < \beta$(2.4)

From above we can easily obtain$-\frac{z\left[D^m_{\varrho\tau\varsigma\xi p}\ f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p}\ f(z)} \prec 2p(1-\eta)\frac{1-\langle 4(1-\eta)\propto-1\rangle z\beta z}{1-z}$ or $\frac{z\left[D^m_{\varrho\tau\varsigma\xi p}\ f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p}\ f(z)} \prec \frac{-2p(1-\eta)+2p(1-\eta)\langle 4(1-\eta)\propto-1\rangle 2p(1-\eta)z}{1-z\beta z}$.

(2.5)Where, $\begin{pmatrix} 0 < \varrho \le \frac{1}{2}, 0 \le \xi < 1, \tau \ge 1, 0 \le \varsigma \le 1, \\ 0 \le \eta < 1, 0 \le \propto < 1, 0 < \beta \le 1 \text{ and } m \in N \end{pmatrix}$.Hence the proof.

Remark 2.1 Since$Re\frac{1-\langle 4(1-\eta)\propto-1\rangle\beta z}{1-\beta z} > \propto$ it follows that$-Re\left\{\frac{1}{2p(1-\eta)} \cdot \frac{z\left[D^m_{\varrho\tau\varsigma\xi p}\ f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p}\ f(z)}\right\} > \propto$.Hence$D^m_{\varrho\tau\varsigma\xi p}\ f(z) \in \sum^*_p(\alpha)$.

Structural formula for the class$\sum_{\varrho\tau\varsigma\xi mp}(\propto, 1, \eta)$

Theorem 2.2 A function $f \in \sum_p$ is in the class$\sum_{\varrho\tau\varsigma\xi mp}(\propto,1,\eta)$ ifand only if there exists a probability measure $\mu(x)$ on ∂U such that$f(z)=\left[z^{-p}+\sum_{k=1-p}^{\infty}\frac{z^k}{\Phi_k(\varrho,\tau\varsigma,,\xi,m,p)}\right]*\left\{z^{-p}\cdot exp\int_{|x|}2p(1-\eta)[1+\langle 4(\eta-1)\propto+1\rangle]\times log(1-xz)\,d\mu(x)\right\}$Where $(z\in U^*)$.(2.6)The correspondence between $\sum_{\varrho\tau\varsigma\xi mp}(\propto,1,\eta)$and the probabilitymeasureμ(x) is one-to-one.

Proof: In view of the subordination condition (2.5), we have that $f\in\int_{\varrho\tau\xi mp}(\propto,1,\eta)$if and only if$-\frac{1}{2p(1-\eta)}\cdot\frac{z\left[D^m_{\varrho\tau\varsigma\xi p}f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p}f(z)}\in P(\alpha)$.From Lemma 1.2, we have$-\frac{z\left[D^m_{\varrho\tau\varsigma\xi p}f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p}f(z)}=2p(1-\eta)\int_{|x|}\frac{1-\langle 4(1-\eta)\propto-1\rangle xz}{1-xz}d\mu(x)$

Which is equivalent to

$\frac{z\left[D^m_{\varrho\tau\varsigma\xi p}f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p}f(z)}=\int_{|x|}\frac{-2p(1-\eta)+2p(1-\eta)\langle 4(1-\eta)\propto-1\rangle xz}{1-xz}d\mu(x)$Integrating this equality, we obtain $z^pD^m_{\varrho\tau\varsigma\xi p}f(z)=exp\int_{|x|}2p(1-\eta)[1+$ *4η−1∝+1log1−xzdμx*or

$D^m_{\varrho\tau\varsigma\xi p}f(z)=z^{-p}\cdot exp\int_{|x|}2p(1-\eta)[1+\langle 4(\eta-1)\propto+1\rangle]\times log(1-xz)\,d\mu(x)$ (2.7)

$D^m_{\varrho\tau\varsigma\xi p}f(z)=(f*h)(z)$.

Where

$h(z)=z^{-p}+\sum_{k=1-p}^{\infty}\Phi_k(\varrho,\tau,\varsigma,\xi,m,p)z^k$

Using (2.10) and above two equations we obtained following result $f(z)=\Big[z^{-p}+$ *k=1−p∞zkΦkϱ,τ,ς,ξ,m,p∗ z−p·exp∫x2p1−η1+4η−1∝+1×log1−xzdμ(x).*

Where $(z\in U^*)$.

Theorem 2.3 Let f $\in\sum_{\varrho\tau\varsigma\xi mp}(\propto,1,\eta)$Then $z^pD^m_{\varrho\tau\varsigma\xi p}f(z)\prec(1-z)^{2p(1-\eta)[1+\langle 4(\eta-1)\propto+1\rangle]}$ $(z\in U)$.Proof: Let $f\in\sum_{\varrho\tau\varsigma\xi mp}(\propto,1,\eta)$Then by (2.5) we have $\frac{z\left[D^m_{\varrho\tau\varsigma\xi p}f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p}f(z)}\prec\frac{-2p(1-\eta)+2p(1-\eta)\langle 4(1-\eta)\propto-1\rangle z}{1-z}$Since the function$\frac{-2p(1-\eta)+2p(1-\eta)\langle 2(1-\eta)\propto-1\rangle z}{1-z}$is univalent and convex in U, inview of Goluzin's result, we obtain$\int_0^z\frac{z\left[D^m_{\varrho\tau\varsigma\xi p}f(\zeta)\right]'}{D^m_{\varrho\tau\varsigma\xi p}f(\zeta)}d\zeta\prec\int_0^z\frac{-2p(1-\eta)+2p(1-\eta)\langle 4(1-\eta)\propto-1\rangle\zeta}{\zeta(1-\zeta)}d\zeta$Or $logD^m_{\varrho\tau\varsigma\xi p}f(z)\prec log\frac{[1-z]^{2p(1-\eta)[1+\langle 4(\eta-1)\propto+1\rangle]}}{(z)^p}$Thus, there exists a function$w\in\Omega$ such that$logD^m_{\varrho\tau\varsigma\xi p}f(z)\prec log\frac{[1-w(z)]^{2p(1-\eta)[1+\langle 4(\eta-1)\propto+1\rangle]}}{w(z)^p}$,which is equivalent to$z^pD^m_{\varrho\tau\varsigma\xi p}f(z)\prec[1-z]^{2p(1-\eta)[1+\langle 4(\eta-1)\propto+1\rangle]}$.

Structural formula for the class$\sum_{\varrho\tau\varsigma\xi mp}(\propto,\beta,\eta)$

Theorem 2.4 Let $f\in\sum_{\varrho\tau\varsigma\xi mp}(\propto,\beta,\eta)$.Then$f(z)=\left[z^{-p}+\sum_{k=1-p}^{\infty}\frac{z^k}{\Phi_k(\varrho,\tau,\varsigma,\xi,m,p)}\right]*\left[z^{-p}exp\left(2p(1-\eta)[\beta+\langle 4(\eta-1)\beta\propto+\beta\rangle]\int_0^z\frac{w(\zeta)}{[1-\beta w(\zeta)]}d\zeta\right)\right]$.

(2.8)Where$(z\in U^*)$ and $(w\in\Omega)$.

Proof: Let$f\in\sum_{\varrho\tau\varsigma\xi mp}(\propto,\beta,\eta)$ and since we have obtained $\frac{z\left[D^m_{\varrho\tau\varsigma\xi p}f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p}f(z)}\prec\frac{p\langle 4(1-\eta)\propto-1\rangle\beta z-p}{1-\beta z}$ $(z\in U),\therefore\frac{z\left[D^m_{\varrho\tau\varsigma\xi p}f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p}f(z)}=\frac{p\langle 4(1-\eta)\propto-1\rangle\beta w(z)-2p(1-\eta)}{1-\beta w(z)}(z\in U)$(2.9)From above equation, we have$\frac{z\left[D^m_{\varrho\tau\varsigma\xi p}f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p}f(z)}+\frac{2p(1-\eta)}{z}=\frac{2p(1-\eta)\langle 4(1-\eta)\propto-1\rangle\beta w(z)-2p(1-\eta)}{1-\beta w(z)}$ (z ∈ U*).Integrating above implies that$\frac{log\left[z^pD^m_{\varrho\tau\varsigma\xi p}f(z)\right]}{2p(1-\eta)[1+\langle 4(\eta-1)\propto+1\rangle]}=\beta\int_0^z\frac{w(\zeta)}{[1-\beta w(\zeta)]}d\zeta$. (2.10)

$D^m_{\varrho\tau\varsigma\xi p}f(z)=(f*h)(z)$,where$h(z)=z^{-p}+\sum_{k=1-p}^{\infty}\Phi_k(\varrho,\tau,\varsigma,\xi,m,p)z^k$.

Using (2.10) and above two equations we obtained following result

$$f(z)=\left[z^{-p}+\sum_{k=1-p}^{\infty}\frac{z^k}{\Phi_k(\varrho,\tau,\varsigma,\xi,m,p)}\right]*\left[z^{-p}exp\left(2p(1-\eta)[1+\langle 4(\eta-1)\propto+1\rangle]\beta\int_0^z\frac{w(\zeta)}{[1-\beta w(\zeta)]}d\zeta\right)\right].$$

Theorem 2.5 If $f\in\Sigma_p$belongs to$\Sigma_{\varrho\tau\varsigma\xi mp}(\propto,\beta,\eta)$, then$D^m_{\varrho\tau\varsigma\xi p}f(z)*$

$$\left\{\begin{matrix}\frac{-2p(1-\eta)z^{-p}+(p+1)z^{1-p}}{(1-z)^2}\left(1-\beta e^{i\theta}\right)\\+\frac{z^{-p}}{(1-z)}\left[2p(1-\eta)-2p(1-\eta)\langle 4(1-\eta)\propto-1\rangle\beta e^{i\theta}\right]\end{matrix}\right\}\neq 0$$

(2.11)for $(z\in U^*)$ and$\theta\in(0,2\pi)$.

Proof: Let$f\in\sum_{\varrho\tau\varsigma\xi mp}(\propto,\beta,\eta)$. Then, from (2.1) it follows$-\frac{z\left[D^m_{\varrho\tau\varsigma\xi p}f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p}f(z)}\neq\frac{-2p(1-\eta)\langle 4(1-\eta)\propto-1\rangle\beta e^{i\theta}+p}{1-\beta e^{i\theta}}$ $(z\in U)$, $\theta\in(0,2\pi)$

(2.12)It is easy to see that the condition (2.12) can be written as follows$\left(1-\beta e^{i\theta}\right)z\left[D^m_{\varrho\tau\varsigma\xi p}f(z)\right]^{re^{i\theta}}D^m_{\varrho\tau\varsigma\xi p}f(z)\neq 0$ (2.13)Note that$D^m_{\varrho\tau\varsigma\xi p}f(z)=D^m_{\varrho\tau\varsigma\xi p}f(z)*\left(z^{-p}+z^{1-p}+\cdots+\frac{1}{z}+1+\frac{z}{1-z}\right)=D^m_{\varrho\tau\varsigma\xi p}f(z)*\frac{z^{-p}}{1-z}$

(2.14) And

$z\left[D^m_{\varrho\tau\varsigma\xi p}f(z)\right]'=D^m_{\varrho\tau\varsigma\xi p}f(z)*\left(-pz^{-p}+(1-p)z^{1-p}-\cdots\right.$ *−1z+z(1−z)2=Dϱτςξpmfz∗−pz−p+1+pz1−p(1−z)2*

(2.15) Where,

$$\left(\begin{matrix}0<\varrho\leq\frac{1}{2},0\leq\xi<1,\tau\geq 1,0\leq\varsigma\leq 1,\\0\leq\eta<1,0\leq\propto<1,0<\beta\leq 1\ and\ \ m\in N\end{matrix}\right).$$

By using (2.13), (2.14) and (2.15), we obtained$-\frac{z\left[D^m_{\varrho\tau\varsigma\xi p}f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p}f(z)}\neq\frac{-2p(1-\eta)\langle 4(1-\eta)\propto-1\rangle\beta e^{i\theta}+p}{1-\beta e^{i\theta}}$ $(z\in U)$, $\theta\in(0,2\pi)$

Coefficient estimates:

Theorem 2.6 Let f of the form (1.6) is in the class$\sum_{\varrho\tau\varsigma\xi mp}(\propto,\beta,\eta)$.Then, for $n\geq 3-2p(1-$

$\eta) \frac{|a_n|(n+p)\Phi_n(\varrho,\tau,\varsigma,\xi,m,p)}{2p(1-\eta)[1+\langle 4(\eta-1)\propto+1\rangle]} \leq \beta$(2.16) Where $\Phi_n(\varrho,\tau,\xi,m,p)$ is given by (1.11)

Proof: To prove the coefficient estimates (2.16) we use the method ofClunie and Koegh [4].Let$f \in \sum_{\varrho\tau\varsigma\xi mp}(\propto,\beta,\eta$.We have$\frac{\frac{z\left[D^m_{\varrho\tau\varsigma\xi p} f(z)\right]'}{D^m_{\varrho\varsigma\tau\xi p} f(z)}+2p(1-\eta)}{\frac{z\left[D^m_{\varrho\tau\varsigma\xi p} f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p} f(z)}+2p(1-\eta)\langle 4(1-\eta)\propto-1\rangle} = zw(z)$where w is analytic in U and $|w(z)| \leq \beta$for $z \in U$.

Then$z\left[D^m_{\varrho\tau\varsigma\xi p} f(z)\right]' + pD^m_{\varrho\tau\varsigma\xi p} f(z) = zw(z)\begin{bmatrix} z\left(D^m_{\varrho\tau\varsigma\xi p} f(z)\right)' \\ +2p(1-\eta)\langle 4(1-\eta)\propto-1\rangle D^m_{\varrho\tau\varsigma\xi p} f(z)\end{bmatrix}$.

(2.17)$zw(z) = \sum_{k=1}^{\infty} w_k z^k$,making use of (1.10) and (2.17), we obtain $\sum_{k=1-p}^{\infty}(k+p)\Phi_k(\varrho,\tau,\varsigma,\xi,m,p)a_k z^{k+p} = \left\{\frac{-p[1+\langle 4(\eta-1)\propto+1\rangle]}{\frac{1}{\sum_{k=1-p}^{\infty}[k+p\langle 4(1-\eta)\propto-1\rangle\Phi_k(\varrho,\tau,\varsigma,\xi,m,p)]a_k z^{k+p}}}\right\}\sum_{k=1}^{\infty} w_k z^k$. (2.18)

Equating the coefficients in (2.18), we have$\Phi_{n-p}(\varrho,\tau,\varsigma,\xi,m,p)a_{n-p} = -2p(1-\eta)[1+\langle 4(\eta-1)\propto+1\rangle]w_n$,

for n = 1,2And$n\Phi_{n-p}(\varrho,\tau,\varsigma,\xi,m,p)a_{n-p} = -2p(1-\eta)[1+\langle 4(\eta-1)\propto+1\rangle]w_n + \sum_{k=1-p}^{n-1-p}[k+$ *2p1−η21−η∝−1Φkakwn−p−k* (2.19)for*n ≥ 3.*

From (2.19), we obtain$\left\{\begin{matrix} -4p(1-\eta)(1-\alpha) + \\ \sum_{k=1-p}^{n-1-p}[k+2p(1-\eta)\langle 4(1-\eta)\propto-1\rangle\Phi_k]a_k z^{k+p}\end{matrix}\right\} \times \sum_{k=1}^{\infty} w_k z^k = \sum_{k=1-p}^{n-p}[(k+p)\Phi_k(\varrho,\tau,\varsigma,\xi,m,p)]a_k z^{k+p} + \sum_{k=1-p}^{n-1-p} c_k z^{k+p}$ (2.20)It is known that, if $h(z) = \sum_{k=1-p}^{n-1-p} h_n z^n$ is analytic in U, then for $0 < r < 1\sum_{n=0}^{\infty}|h_n|^2 r^{2n} = \frac{1}{2\pi}\int_0^{2\pi}\left|h(re^{ii\theta})\right|^2 d\theta$.(2.21)

Since $\left|\sum_{k=1}^{\infty} w_k z^k\right| < \beta|z| < \beta$, Making use of (2.20) and (2.21), we have$\sum_{k=1-p}^{n-p}\left[\left(k+\right.\right.$ *2p1−η2Φkϱ,τ,ς,ξ,m,p2ak2r2k+p*

$+\sum_{k=n+1-p}^{\infty}|c_k|^2 r^{2(k+p)} \leq \beta^2 \times \left\{\begin{matrix} p^2[1+\langle 4(\eta-1)\propto+1\rangle]^2 + \\ \sum_{k=1-p}^{n-1-p}\begin{matrix}[k+2p(1-\eta)\langle 4(1-\eta)\propto-1\rangle]^2 \\ \times\left(\Phi_k(\varrho,\tau,\varsigma,\xi,m,p)\right)^2|a_k|^2 r^{2(k+p)}\end{matrix}\end{matrix}\right\}$.

Letting $r \to 1$, we obtain $\sum_{k=1-p}^{n-1-p}[k+2p(1-\eta)]^2\left(\Phi_k(\varrho,\tau,\varsigma,\xi,m,p)\right)^2|a_k|^2 \leq 4(1-\eta)^2p^2\beta^2[1+\langle 4(\eta-1)\propto+1\rangle]^2 + \sum_{k=1-p}^{n-1-p}[k+2p(1-\eta)\langle 4(1-\eta)\propto-1\rangle]^2\left(\Phi_k(\varrho,\tau,\varsigma,\xi,m,p)\right)^2|a_k|^2$.The above inequality implies$n^2\Phi_{n-p}(\varrho,\tau,\varsigma,\xi,m,p)^2\left|a_{n-p}\right|^2 \leq 4(1-\eta)^2p^2\beta^2[1+\langle 4(\eta-1)\propto+1\rangle]^2 + \sum_{k=1-p}^{n-1-p}[k+2p(1-\eta)]^2\left(\Phi_k(\varrho,\tau,\varsigma,\xi,m,p)\right)^2|a_k|^2$Finally, replacing n − p by n, we have$|a_n| \leq \frac{4p(1-\eta)\beta\left[\frac{1+\langle 4(\eta-1)\propto+1\rangle}{2}\right]}{(n+2p(1-\eta))\Phi_k(\varrho,\tau,\varsigma,\xi,m,p)}$,

Where,

$\left(\begin{matrix} 0 < \varrho \leq \frac{1}{2}, 0 \leq \xi < 1, \tau \geq 1, 0 \leq \varsigma \leq 1, \\ 0 \leq \eta < 1, 0 \leq \propto < 1, 0 < \beta \leq 1 \; and \; m \in N\end{matrix}\right)$.Thus, the proof of our theorem is completed.Theorem 2.6 enables us to obtain a distortion result for the class$\sum_{\varrho\tau\varsigma\xi mp}(\propto,\beta,\eta)$.

Corollary 2.1: If $f \in \sum_{\varrho\tau\varsigma\xi mp}(\propto,\beta,\eta)$ is given by (1.6), then for$0 < |z| = r < 1|f(z)| \leq \frac{1}{r^p} + p\beta[1+\langle 4(\eta-1)\propto+1\rangle] \times r^{1-p}\sum_{k=1-p}^{\infty}\frac{1}{(k+p)\Phi_k(\varrho,\tau\varsigma,\xi,m,p)}.|f(z)| \geq \frac{1}{r^p} + 2p(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle] \times r^{1-p}\sum_{k=1-p}^{\infty}\frac{1}{(k+2p(1-\eta))\Phi_k(\varrho,\tau,\varsigma,\xi,m,p)}$.And $|f'(z)| \geq \frac{2p(1-\eta)}{r^{p+1}} - 2p(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle] \times r^{2-p}\sum_{k=1-p}^{\infty}\frac{1}{(k+2p(1-\eta))\Phi_k(\varrho,\tau,\varsigma,\xi,m,p)}.|f'(z)| \leq \frac{2p(1-\eta)}{r^{p+1}} + 4p(1-\eta)^2\beta[1+\langle 4(\eta-1)\propto+1\rangle] \times r^{2-p}\sum_{k=1-p}^{\infty}\frac{1}{(k+2p(1-\eta))\Phi_k(\varrho,\tau,\varsigma,\xi,m,p)}$.In the sequence we give a sufficient condition for a function tobelong to the class$\sum_{\varrho\tau\varsigma\xi mp}(\propto,\beta,\eta)$.

Theorem 2.7 Let $f \in \sum_p$be given by (1.6). If for $0 \leq \alpha < 1$and$0 < \beta \leq 1$ $\sum_{k=1-p}^{\infty}\{k(\beta+1)+2p(1-\eta)[1+$ *β41−η∝−1×Φkϱ,τ,ξ,m,pak≤p1−ηβ1+4η−1∝+1*(2.22)*f∈ ∑ϱτςξmp(∝,β,η).*

Proof: Assume that$f(z) = z^{-p} + \sum_{k=1-p}^{\infty} a_k z^k$. We have$M = \left|z\left[D^m_{\varrho\tau\varsigma\xi p} f(z)\right]' + 2p(1-\eta)D^m_{\varrho\tau\varsigma\xi p} f(z)\right| - \beta\left|z\left[D^m_{\varrho\tau\varsigma\xi p} f(z)\right]' + 2p(1-\eta)\langle 4(1-\eta)\propto-1\rangle D^m_{\varrho\tau\varsigma\xi p} f(z)\right| = \left|\sum_{k=1-p}^{\infty}(k+2p(1-\eta))\Phi_k(\varrho,\tau,\varsigma,\xi,m,p)a_k z^k\right| - \beta\left|\frac{-2p(1-\eta)[1+\langle 4(\eta-1)\propto+1\rangle]}{z^p}\right| + \sum_{k=1-p}^{\infty}[k+2p(1-$ *η41−η∝−1Φkϱ,τ,ς,ξ,m,pakzk*

For$0 < |z| = r < 1$, we obtain$r^pM \leq \sum_{k=1-p}^{\infty}(k+$ *pΦkϱ,τ,ς,ξ,m,pakrk+p−* $\beta\begin{bmatrix} 2p(\eta-1)(\langle 4(\eta-1)\propto+1\rangle+1) \\ -\sum_{k=1-p}^{\infty}|k+2p(1-\eta)\langle 4(1-\eta)\propto-1\rangle|\Phi_k a_k r^{k+p}\end{bmatrix}$. *Or* $r^pM \leq \sum_{k=1-p}^{\infty}\{k(\beta+1)+2p(1-\eta)[1+\beta\langle 4(1-\eta)\propto-1\rangle]\} \times$ Φkakrk+p−2p1−ηβ1+4η−1∝+1.Since the above inequality holds for all r $(0 < r < 1)$, letting $r \to 1$,we have$M \leq \sum_{k=1-p}^{\infty}\left\{\begin{matrix} k(\beta+1)+2p(1-\eta) \\ \times[1+\beta\langle 4(1-\eta)\propto-1\rangle]\end{matrix}\right\} \times \Phi_k|a_k| - 2p(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]$.Making use of (2.22), we obtain $M \leq 0$,that is$\left|\frac{z\left[D^m_{\varrho\tau\varsigma\xi p} f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p} f(z)} + 2p(1-\eta)\right| < \beta\left|\frac{z\left[D^m_{\varrho\tau\varsigma\xi p} f(z)\right]'}{D^m_{\varrho\tau\varsigma\xi p} f(z)} + 2p(1-\eta)\langle 4(1-\eta)\propto\right.$ *−1.*Consequently,*f∈∑ϱτςξmp(∝,β,η).*Where,

$\left(\begin{matrix} 0 < \varrho \leq \frac{1}{2}, 0 \leq \xi < 1, \tau \geq 1, 0 \leq \varsigma \leq 1, \\ 0 \leq \eta < 1, 0 \leq \propto < 1, 0 < \beta \leq 1 \; and \; m \in N\end{matrix}\right)$.

3 *Properties of the class*$\sum^+_{\varrho\tau\varsigma\xi mp}(\propto,\beta,\eta)$

We begin this section by proving that the condition (2.22) is bothnecessary and sufficient for a function to be in the class$\sum^+_{\varrho\tau\varsigma\xi mp}(\propto,\beta,\eta)$.

Theorem 3.1 Let $f \in \sum_p^+$. Then f belongs to the class $\sum_{\varrho\tau\varsigma\xi mp}^+ (\propto,\beta,\eta)$if and only if$\sum_{k=1-p}^{\infty}\{k(\beta+1)+$ *2p1−η1+β21−η∝−1×Φkϱ,τ,ς,ξ,m,pak≤2p1−ηβ1+4η−1∝+1.*

Proof: In view of Theorem 2.7, we have to prove "only if" part. Assume that $f(z)=z^{-p}+\sum_{k=1-p}^{\infty} a_k z^k \quad (a_k \geq 0,$ $p\in N$)is in the class$\sum_{\varrho\tau\varsigma\xi mp}^+ (\propto,\beta,\eta)$.Then $\left|\frac{\frac{z\left[D_{\varrho\tau\varsigma\xi p}^m f(z)\right]'}{D_{\varrho\tau\varsigma\xi p}^m f(z)}+2p(1-\eta)}{\frac{z\left[D_{\varrho\tau\varsigma\xi p}^m f(z)\right]'}{D_{\varrho\tau\varsigma\xi p}^m f(z)}+2p(1-\eta)\langle 4(1-\eta)\propto-1\rangle}\right| = \left|\frac{\sum_{k=1-p}^{\infty}(k+2p(1-\eta))\Phi_k(\varrho,\tau,\varsigma,\xi,m,p)a_k r^{k+p}}{\frac{2p(1-\eta)[1+\langle 4(\eta-1)\propto+1\rangle]}{z^p}-\sum_{k=1-p}^{\infty}[k+2p(1-\eta)\langle 4(1-\eta)\propto-1\rangle]\Phi_k a_k z^k}\right| < \beta$for all $z \in U$. Since Re z ≤ |z|for all z, it follows that

$$\mathrm{Re}\left\{\frac{\sum_{k=1-p}^{\infty}(k+2p(1-\eta))\Phi_k(\varrho,\tau,\varsigma,\xi,m,p)a_k r^{k+p}}{\frac{2p(1-\eta)[1+\langle 4(\eta-1)\propto+1\rangle]}{z^p}-\sum_{k=1-p}^{\infty}[k+2p(1-\eta)\langle 4(1-\eta)\propto-1\rangle]\Phi_k a_k z^k}\right\} < \beta. \qquad (3.1)$$

We choose the values of z on the real axis such that$\frac{1}{2p(1-\eta)}\cdot\frac{z\left[D_{\varrho\tau\varsigma\xi p}^m f(z)\right]'}{D_{\varrho\tau\varsigma\xi p}^m f(z)}$is real. Upon clearing the denominator in (3.1) and letting $z\to 1$through positive values, we obtain$\sum_{k=1-p}^{\infty}(k+2p(1-\eta))\Phi_k(\varrho,\tau,\varsigma,\xi,m,p)a_k \leq 2p(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]-\sum_{k=1-p}^{\infty}\beta\,[k+$ *2p1−η41−η∝−1Φkϱ,τ,ς,ξ,m,pak*

Or

$\sum_{k=1-p}^{\infty}\{k(\beta+1)+2p(1-\eta)[1+\beta\langle 4(1-\eta)\propto$ *−1)Φkϱ,τ,ς,ξ,m,pak≤pβ1+4η−1∝+1.*

Where,

$$\begin{pmatrix} 0<\varrho\leq\frac{1}{2}, 0\leq\xi<1, \tau\geq 1, 0\leq\varsigma\leq 1, \\ 0\leq\eta<1, 0\leq\propto<1, 0<\beta\leq 1 \text{ and } m\in N\end{pmatrix}$$

.Hence, the result follows.

Corollary 3.1: Iff $\in \sum_p^+$ given by (1.7) is in the class $\sum_{\varrho\tau\varsigma\xi mp}^+ (\propto,\beta,\eta)$thena$_n \leq$

$$\frac{2p(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]}{\{n(\beta+1)+2p(1-\eta)[1+\beta\langle 4(1-\eta)\propto-1\rangle]\}\Phi_k(\varrho,\tau,\varsigma,\xi,m,p)},\quad n\geq 1-2p(1-\eta). \qquad (3.2)$$

with equality for the functions of the form$f_n(z)=\frac{1}{z^p}-\frac{2p(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]}{\{n(\beta+1)+2p(1-\eta)[1+\beta\langle 4(1-\eta)\propto-1\rangle]\}\Phi_k(\varrho,\tau,\varsigma,\xi,m,p)}z^n$Coefficient estimates obtained in Corollary 3.1 enables us to givea distortion result for the class$\sum_{\varrho\tau\varsigma\xi mp}^+ (\propto,\beta,\eta)$.

Theorem 3.2 If $f\in\sum_{\varrho\tau\varsigma\xi mp}^+ (\propto,\beta,\eta)$,then for $0<|z|=r<1$ $|f(z)|\geq$

$$\frac{1}{z^p}-\frac{2p(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]}{\{\beta[1-2p(1-\eta)+\langle 4(1-\eta)\propto-1\rangle 2p(1-\eta)]+1\}\Phi_{1-p}(\varrho,\tau,\varsigma,\xi,m,p)}r^{1-p}$$

And $|f(z)|\geq$

$$\frac{1}{z^p}+\frac{2p(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]}{\{\beta[1-2p(1-\eta)+\langle 4(1-\eta)\propto-1\rangle 2p(1-\eta)]+1\}\Phi_{1-p}(\varrho,\tau,\varsigma,\xi,m,p)}r^{1-p}$$

where equality holds for the function$f_p(z)=\frac{1}{z^p}+\frac{2p(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]}{\{\beta[1-2p(1-\eta)+\langle 4(1-\eta)\propto-1\rangle 2p(1-\eta)]+1\}\Phi_{1-p}(\varrho,\tau,\varsigma,\xi,m,p)}z^{1-p}$at $z=ir,r.$

Proof: Suppose $f\in\sum_{\varrho\tau\varsigma\xi mp}^+ (\propto,\beta,\eta)$,Making use of inequality$\sum_{k=1-p}^{\infty} a_k \leq$

$$\frac{2p(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]}{\{\beta[1-2p(1-\eta)+\langle 4(1-\eta)\propto-1\rangle 2p(1-\eta)]+1\}\Phi_{1-p}(\varrho,\tau,\varsigma,\xi,m,p)} \qquad (3.3)$$

which follows easily from Theorem 3.1, the proof is trivial.Now, we prove that the class$\sum_{\varrho\tau\varsigma\xi\,\mathrm{mp}}^+(\propto,\beta,\eta)$ is closed under convolution.

Theorem 3.3 Let $h(z)=z^{-p}+\sum_{k=1-p}^{\infty} c_k z^k$ be analytic in U^* and$0\leq c_k\leq 1$. If f given by (1.7) is in the class $\sum_{\varrho\tau\varsigma\xi\,\mathrm{mp}}^+(\propto$,β,η then f ∗ his also in the class*ϱτςξmp+∝,β,η.*

Proof: Sincef $\in\sum_{\varrho\tau\xi\,\mathrm{mp}}^+(\propto,\beta,\eta)$, then by Theorem 3.1, we have$\sum_{k=1-p}^{\infty}\{k(\beta+1)+$ *2p1−η1+β41−η∝−1Φkϱ,τ,ς,ξ,m,pak≤2p1−ηβ1+4η−1∝+1.*In view of the above inequality and the fact that$(f*h)(z)=z^{-p}+\sum_{k=1-p}^{\infty} a_k c_k z^k$we obtain$\sum_{k=1-p}^{\infty}\{k(\beta+1)+2p(1-\eta)[1+\beta\langle 4(1-\eta)\propto$ *−1Φkϱ,τ,ς,ξ,m,pakck≤k=1−p∞kβ+1+2p1−η1+β41−η∝−1Φkϱ,τ,ς,ξ,m,pak≤2p1−ηβ1+4η−1∝+1.*Therefore, by Theorem 3.1, the result follows.The next result involves an integral operator which was investigatedin many papers [2], [6], [20].

Theorem 3.4 If $f\in\sum_{\varrho\tau\varsigma\xi mp}^+ (\propto,\beta,\eta)$, then the integral operator

$$F_{c,p}(z)=\frac{c}{z^{p+c}}\int_0^z t^{c+2p(1-\eta)+1} f(t)dt\,, \qquad c>0.$$

It is also in the class$\sum_{\varrho\tau\varsigma\xi mp}^+ (\propto,\beta,\eta)$.

Proof: It is easy to check that

$$F_{c,p}(z)=f(z)*\left(z^{-p}+\sum_{k=1-p}^{\infty}\frac{c}{c+2p(1-\eta)+k}z^k\right).$$

Since $0<\frac{c}{c+2p(1-\eta)+k}\leq 1$,by Theorem 3.3, the proof is trivial.Where,

$$\begin{pmatrix} 0<\varrho\leq\frac{1}{2}, 0\leq\xi<1, 0\leq\varsigma\leq 1, \tau\geq 1, \\ 0\leq\eta<1, 0\leq\propto<1, 0<\beta\leq 1 \text{ and } m\in N\end{pmatrix}.$$

4 Neighborhoods and partial sums

Following earlier investigations on the familiar concept of neighborhoodsof analytic functions by Goodman [7], Ruschweyh [17] and morerecently by Liu and Srivastava [9], [10], Liu [11], Altinta¸s et al. [1],Orhan and Kamali [14], Srivastava and Orhan [19], Orhan [15], Denizand Orhan [5] and Aouf [3], we define the *(n, δ)*− neighborhood of afunction $f\in\sum_p$of the form (1.6) as follows.

Definition 4.1 For $\delta=\frac{2(\tau-\xi)^2+\frac{1}{\varrho}(\tau-\xi\ )(1-\varrho^2)}{1+2(\tau-\xi)^2+\frac{1}{\varrho}(\tau-\xi\ )(1-\varrho^2)}>0$ and nonegative sequence $S=\{s_k\}_{k=1-p}^{\infty}$Where $s_k=$

$$\frac{\{\beta[1-2p(1-\eta)+|\langle 4(1-\eta)\propto-1\rangle|2p(1-\eta)]+1\}\Phi_{1-p}(\varrho,\tau,\varsigma,\xi,m,p)}{2p(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]} \qquad (4.1)$$

$(k\geq 1-p, p\in N, 0\leq\alpha<$ *1, 0<β≤1.*The *(n, δ)*−neighborhood of a function *f∈∑p* of the form (1.6) isdefined by$N_\delta(f)=\left\{\begin{matrix} g\in\sum_p : g(z)=z^{-p}+\sum_{k=1-p}^{\infty} b_k z^k \text{ and} \\ \sum_{k=1-p}^{\infty} s_k|b_k-a_k|\leq\delta\end{matrix}\right\}.$ (4.2)For$s_k=k$, Definition 1.4 corresponds to the (n,δ)−neighborhoodconsidered by Ruscheweyh [17].Making use of Definition 4.1, we prove the first result on (n,δ)-neighborhood of the class$\sum_{\varrho\tau\varsigma\xi mp}(\propto,\beta,\eta)$.

Theorem 4.1 Let $f \in \sum_{\varrho\tau\varsigma\xi mp}(\propto,\beta,\eta)$ be given by (1.6). If f satisfies$[f(z)+\varepsilon z^p](1+\varepsilon)^{-1} \in \sum_{\varrho\tau\varsigma\xi mp}(\propto,\beta,\eta)$

(4.3)$(\varepsilon \in C,\ |\varepsilon|<\delta,\ \delta \geq 0)$

then$N_\delta(f) \subset \sum_{\varrho\tau\varsigma\xi mp}(\propto,\beta,\eta)$

(4.4)

Proof: It is not difficult to see that a function$f \in \sum_{\varrho\tau\varsigma\xi mp}(\propto,\beta,\eta)$if and only if$\frac{z\left[D^m_{\varrho\tau\varsigma\xi p}f(z)\right]'+2p(1-\eta)D^m_{\varrho\tau\varsigma\xi p}f(z)}{\beta z\left[D^m_{\varrho\tau\varsigma\xi p}f(z)\right]'+\beta\langle 4(1-\eta)\propto-1\rangle 2p(1-\eta)D^m_{\varrho\tau\varsigma\xi p}f(z)} \neq \sigma$(4.5)

$(z \in U,\ \sigma \in C,\ |\sigma|=1)$ which is equivalent to$\frac{(f*h)(z)}{z^{-p}} \neq 0 \qquad (z \in U).$ (4.6)Where for convenience,$h(z)=z^{-p}+\sum_{k=1-p}^{\infty} c_k z^k = z^{-p}+\sum_{k=1-p}^{\infty}\frac{\{\beta\sigma[k+\langle 4(1-\eta)\propto-1\rangle 2p(1-\eta)]-(k+2p(1-\eta))\}\Phi_k(\varrho,\tau,\varsigma,\xi,m,p)}{2p(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]\sigma}z^k$

(4.7)From (4.7) it follows that$|c_k| = \left|\frac{\{\beta\sigma[k+\langle 4(1-\eta)\propto-1\rangle 2p(1-\eta)]-(k+2p(1-\eta))\}\Phi_k(\varrho,\tau,\varsigma,\xi,m,p)}{2p(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]\sigma}\right| \leq \frac{\{\beta\sigma[k+|\langle 4(1-\eta)\propto-1\rangle|2p(1-\eta)]+(k+2p(1-\eta))\}\Phi_k(\varrho,\tau,\varsigma,\xi,m,p)}{2p(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]\sigma}(k \geq 1-p,$ $p \in N.$ (4.8)Furthermore, under the hypotheses (4.3), using (4.6) we obtain thefollowing assertions:$\frac{[f(z)+\varepsilon z^p](1+\varepsilon)^{-1}*h(z)}{z^{-p}} \neq 0.Or\frac{(f*h)(z)}{z^{-p}} \neq \varepsilon$

$(z \in U)$.which is equivalent$\left|\frac{(f*h)(z)}{z^{-p}}\right| \geq \delta$(4.9)Now, if we let$g(z)=z^{-p}+\sum_{k=1-p}^{\infty} b_k z^k \in N_\delta(f)$,then we have$\left|\frac{[f(z)-g(z)]*(z)}{z^{-p}}\right| = \left|\sum_{k=1-p}^{\infty}(a_k-b_k)z^{p+k}\right| \leq \sum_{k=1-p}^{\infty}\frac{\{\beta[k+|\langle 4(1-\eta)\propto-1\rangle|2p(1-\eta)]+(k+2p(1-\eta))\}\Phi_k(\varrho,\tau,\varsigma,\xi,m,p)}{2p(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]}|a_k-$ $bkzk+p<\delta z\in U,\ k\geq 1-p,\ p\in N,\ \delta>0.$

Where,

$\left(\begin{matrix}0<\varrho\leq\frac{1}{2},0\leq\xi<1,0\leq\varsigma\leq 1\\ ,\tau\geq 1,0\leq\eta<1,0\leq\propto<1,0<\beta\leq 1\ and\ m\in N\end{matrix}\right)$.Thus, for any complex number σ with $|\sigma|=1$, we have$\frac{(g*h)(z)}{z^{-p}} \neq 0$

$(z \in U)$which implies$g \in \sum_{\varrho\tau\varsigma\xi mp}(\propto,\beta,\eta)$. The proof of the theorem is completed.In the sequence we give the definition of (n,δ) −neighborhood of afunction $f \in \sum_p^+$ of the form (1.7).

Definition 4.2 For $\delta>0$ and a non-negative sequence$S=\{s_k\}_{k=1-p}^{\infty}$where$s_k = \frac{\{k(\beta+1)+2p(1-\eta)[1+\beta\langle 4(1-\eta)\propto-1\rangle]\}\Phi_k(\varrho,\tau,\varsigma,\xi,m,p)}{2p(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]}(k \geq 1-p, p \in$ $N,\ 0\leq\propto<1,\ 0<\beta\leq 1$The (n,δ)−neighborhood of a function $f \in \sum_p^+$ of the form (1.7) isdefined by$N_\delta(f)=$

$\left\{\begin{matrix}g \in \sum_p^+: g(z)=z^{-p}+\sum_{k=1-p}^{\infty} b_k z^k\\ and\ \sum_{k=1-p}^{\infty} s_k|b_k-a_k| \leq \delta\end{matrix}\right\}$(4.10)We have the following result on (n,δ) −neighborhood of the class$\sum^+_{\varrho\tau\varsigma\xi mp}(\alpha,\beta,\eta)$.

Theorem 4.2 If the function f given by (1.7) is in the class $\sum^+_{\varrho\tau\varsigma\xi mp}(\alpha,\beta,\eta)$,Then$N_\delta(f) \subset \sum^+_{\varrho\tau\varsigma\xi mp}(\alpha,\beta,\eta)$,

(4.11)Where$\delta = \frac{2(\tau-\xi)(\tau-\varsigma)+\frac{(\tau-\xi)-\varrho^2(\tau-\varsigma)}{\varrho}}{1+2(\tau-\xi)(\tau-\varsigma)+\frac{(\tau-\xi)-\varrho^2(\tau-\varsigma)}{\varrho}}$.

The result is the best possible in the sense that δ cannot be increased.

Proof: For a function $f \in \sum^+_{\varrho\tau\xi mp}(\alpha,\beta,\eta)$ of the form (1.7), Theorem 3.1immediately yields$\sum_{k=1-p}^{\infty}\frac{\{k(\beta+1)+2p(1-\eta)[1+\beta\langle 4(1-\eta)\propto-1\rangle]\}\Phi_k(\varrho,\tau,\varsigma,\xi,m,p)}{2p(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]}a_k \leq \frac{1}{\Phi_{1-p}(\varrho,\tau,\varsigma,\xi,1,p)}$(4.12) Let $g(z)=z^{-p}+\sum_{k=1-p}^{\infty} b_k z^k \in N_\delta(f)$ with $\delta = \frac{2(\tau-\xi)(\tau-\varsigma)+\frac{(\tau-\xi)-\varrho^2(\tau-\varsigma)}{\varrho}}{1+2(\tau-\xi)(\tau-\varsigma)+\frac{(\tau-\xi)-\varrho^2(\tau-\varsigma)}{\varrho}} > 0.$From the condition (4.10) we find that$\sum_{k=1-p}^{\infty} s_k|b_k-a_k| \leq \delta$

(4.13)Using (4.12) and (4.13), we obtain$\sum_{k=1-p}^{\infty} s_k b_k \leq \sum_{k=1-p}^{\infty} s_k a_k+\sum_{k=1-p}^{\infty} s_k|b_k-a_k| \leq \frac{1}{\Phi_{1-p}(\varrho,\tau,\varsigma,\xi,1,p)}+\delta=1$Thus, in view of Theorem 3.1, we get $g \in \sum^+_{\varrho\tau\varsigma\xi mp}(\alpha,\beta,\eta)$.To prove the sharpness of the assertion of the theorem, we considerthe functions $f \in \sum^+_{\varrho\tau\xi mp}(\alpha,\beta,\eta)$ and $f \in \sum_p^+$given by$f(z)=$ $z^{-p}+\frac{2p(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]}{\{\beta[1-2p(1-\eta)+\langle 4(1-\eta)\propto-1\rangle 2p(1-\eta)]+1\}\Phi_{1-p}(\varrho,\tau,\varsigma,\xi,m,p)}z^{1-p}$

(4.14)And$f(z)=$ $z^{-p}+\left[\frac{2p(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]}{\{\beta[1-2p(1-\eta)+\langle 4(1-\eta)\propto-1\rangle 2p(1-\eta)]+1\}\Phi_{1-p}(\varrho,\tau,\varsigma,\xi,m,p)}+\frac{2p(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]}{\{\beta[1-2p(1-\eta)+\langle 4(1-\eta)\propto-1\rangle 2p(1-\eta)]+1\}\Phi_{1-p}(\varrho,\tau,\varsigma,\xi,m,p)}\right]z^{1-p}$

(4.15)Where,

$\left(\begin{matrix}0<\varrho\leq\frac{1}{2},0\leq\varsigma\leq 1,0\leq\xi<1,\tau\geq 1,\\ 0\leq\eta<1,0\leq\propto<1,0<\beta\leq 1\ and\ m\in N\end{matrix}\right)$.Hence the function $g \in N_\delta(f)$but according to Theorem 3.1,$g \in \sum^+_{\varrho\tau\varsigma\xi mp}(\alpha,\beta,\eta)$.Consequently, the proof of our theorem iscompleted.Next, we investigate the ratio of real parts of functions of the form(1.6) and their sequences of partial sums defined by$k_m(z)=$

$\left\{\begin{matrix}z^{-p}, & m=1,2,\cdots,-p\\ z^{-p}+\sum_{k=1-p}^{m-1} a_k z^k & m=1-p,2-p,\cdots\end{matrix}\right.$

(4.16)We also determine sharp lower bounds for $Re\left\{\frac{f(z)}{k_m(z)}\right\}$ and $Re\left\{\frac{k_m(z)}{f(z)}\right\}$.

Theorem 4.3 Let $f \in \sum_p$ be given by (1.6) and let$k_m(z)$ be given by(4.16). Suppose that$\sum_{k=1-p}^{\infty}\theta_k|a_k| \leq 1$ (4.17)Where$\theta_k = \frac{\{k(\beta+1)+2p(1-\eta)[1+\beta\langle 4(1-\eta)\propto-1\rangle]\}\Phi_k(\varrho,\tau,\varsigma,\xi,m,p)}{2p(1-\eta)\beta[1+\langle 4(\eta-1)\propto+1\rangle]}$Then, for $m \geq 1-p$, we have$Re\left\{\frac{f(z)}{k_m(z)}\right\} > 1-\frac{1}{\theta_m}$

(4.18)And$Re\left\{\frac{k_m(z)}{f(z)}\right\} > \frac{\theta_m}{1+\theta_m}$

(4.19)The results are sharp for each $m \geq 1-p$ with the extremal functiongiven by$f(z)=z^{-p}-\frac{1}{\theta_m}z^m$

(4.20)

Proof:

Under the hypotheses of the theorem, we can see from (4.17) that$\theta_{k+1} > \theta_k > 1 \qquad (k>1-p)$.Therefore, by using hypotheses (4.17) again, we have$\sum_{k=1-p}^{m-1}|a_k|+$

$\theta_m \sum_{k=m}^{\infty} |a_k| \le \sum_{k=1-p}^{\infty} \theta_k |a_k| \le 1.$ (4.21)Let$w(z) = \theta_m \left[\frac{f(z)}{k_m(z)} - \left(1 - \frac{1}{\theta_m}\right)\right] = 1 + \frac{\theta_m \sum_{k=m}^{\infty} a_k z^{k+p}}{1+\sum_{k=1-p}^{m-1} a_k z^{k+p}}$ (4.22)Applying (4.21) and (4.22), we find$\left|\frac{w(z)-1}{w(z)+1}\right| = \left|\frac{\theta_m \sum_{k=m}^{\infty} a_k z^{k+p}}{2+2\sum_{k=1-p}^{m-1} a_k z^{k+p} + \theta_m \sum_{k=m}^{\infty} a_k z^{k+p}}\right| \le \frac{\theta_m \sum_{k=1-p}^{m-1} |a_k|}{2-2\sum_{k=1-p}^{m-1} |a_k| - \theta_m \sum_{k=1-p}^{m-1} |a_k|} \le 1 (z) \in U$ (4.23)From above it is clear that Re $w(z) > 0$, $(z \in U)$.From (4.22), we immediatelyobtain (4.18).To prove that the function *f* defined by (4.20) gives sharp result,we can see that for $z \to 1 - \frac{f(z)}{k_m(z)} = 1 - \frac{1}{\theta_m} z^m \to 1 - \frac{1}{\theta_m}$which shows that the bound in (4.18) is the best possible.Similarly, if we let$\Phi_z = (1+\theta_m)\left[\frac{k_m(z)}{f(z)} - \frac{\theta_m}{1+\theta_m}\right] = 1 - \frac{(1+\theta_m)\sum_{k=m}^{\infty} a_k z^{k+p}}{1+\sum_{k=1-p}^{\infty} a_k z^{k+p}}$(4.24)and making use of (4.21), we find that$\left|\frac{\Phi(z)-1}{\Phi(z)+1}\right| = \left|\frac{-(1+\theta_m)\sum_{k=m}^{\infty} a_k z^{k+p}}{2+2\sum_{k=1-p}^{\infty} a_k z^{k+p} - (1+\theta_m)\sum_{k=m}^{\infty} a_k z^{k+p}}\right| \le \frac{(1+\theta_m)\sum_{k=m}^{\infty} |a_k|}{2-2\sum_{k=1-p}^{\infty} |a_k| + (1+\theta_m)\sum_{k=m}^{\infty} |a_k|} \le 1$ (4.25)

Where,

$\begin{pmatrix} 0 < \varrho \le \frac{1}{2}, 0 \le \varsigma \le 1, 0 \le \xi < 1, \tau \ge 1, \\ 0 \le \eta < 1, 0 \le \propto < 1, 0 < \beta \le 1 \; and \; m \in N \end{pmatrix}$.which leads immediately to the assertion (4.19) of the theorem.The bound in (4.19) is sharp for each m ≥ 1−p, with the extremalfunction f given by (4.20). The proof of the theorem is now completed.

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