# Relative w-projective and dimension

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***Abstract-*** In this paper we generalize the notion of projective, injective and flat modules and dimension. Hence, we introduce and study the notion of Quasi –w projective modules and dimension.

***Index Terms-*** projective, injective and flat module with dimension, Quasi –w projective

## I. INTRODUCTION

Throughout this paper , all rings are associative and all modules – if not specified otherwise are left and unitary. Let R is a ring and M be an R-module as usual we use pdR(M) , idR(M) and fdR(M) to denote respectively. The classical projective dimension, injective dimension and flat dimension of M. we use also gldim(R) and wdim(R) to denote respectively , the classical global and Quasi w dimension of R. The character module Homz(M,Q/Z) is denoted by M1.

Recall that a ring R is called left perfect if every flat module is projective.

Example of perfect rings includes Quasi-Frobenius rings or left Artinian. It is shown that a ring is Quasi-Frobenius if and only if every projective module is injective if and only if every injective module is projective. We introduce and study a new generalization of projective and injective modules and dimension. The relation between the Quasi-w projective dimension and other dimension are discussed.

Definition 1.1:- For an R-module M the Quasi w-projective dimension of M qwpdR(M) is defined to be the smallest integer n $\geq 0$ such that $Ext_R^{n+1}(M,M)$= 0 for all flat modules M. If no such integer exist set qwpdR (M) = $\infty$. If qwpdR(M) = 0 then M will be called a Quasi w-projective module.

Example 1.2:- Consider the local Quasi-Frobenius ring R= $\frac{m[Z]}{Z^2}$ where m is a field and denote by $\bar{Z}$ the residue class in R of Z, then $\bar{Z}$ is a Quasi –w projective R-module which is not projective.

Proof :- Since R is Quasi –Frobenius, every projective and every flat since R is perfect modules M is injective then $Ext_R^i((\overline{Z}\,, M))$= 0 for all I >0. Thus $\bar{Z}$ is a Quasi –w-projective R-module Now, if we suppose that $\bar{Z}$ is projective then it must be free since R is local a contradiction since $\bar{Z}^2$ = 0. So we conclude that $\bar{Z}$ is not projective as desired.

In [4] the authors defined and studied a refinement of flat Modules which they called the IF modules. Recall that an R-module if $Tor_R^i(I,M)$ = 0 for all right injective R-module I and all I > 0.

Proposition 1.3 :- Let R be a right coherent ring then every Quasi-w projective R-module is an IF R-module.
Proof :- Let M be a Quasi –w projective R-module. Let E be an injective right R-module then $\bar{E}$ is flat then $Ext_R^i(M\,,\bar{E})$ =0 for all i > 0. While $Ext_R^i(M\,,\bar{E})$ =( $Tor_R^i(\bar{E}\,,M)$ ). Hence $Tor_R^i$ ( E, M) = 0 thus M is an IF- module.

Proposition 1.4 :- Let M be a Quasi –w projective R-module then

i) $Ext_R^i(M,M^*)$ = 0 for all i > 0 and all M* with finite flat dimension.
ii) $Either$ M is Projective or $fd_R(M)=\infty$

Proof :-

i) Since $Ext_R^i(M,M^*)$=0 for all flat modules M* and all i >0, the proof is immediate by dimension shifting.

ii) Suppose that $fd_R(M) <\infty$ and pick a short exact sequence 0→M*→P→M→$\infty$ where P is projective. Clearly $fd_R(M^*)<\infty$ , then $Ext_R^1(M,M^*)$ = 0 thus the short exact sequence splits and so M is isomorphic to a direct summand of P and then projective.

Corollary 1.5 :- A module M is Quasi –projective if and only if it is flat and Quasi –w projective.
Proof :- Let M be an R-module. The cotorsion dimension of M $cd_R(M)$ is smallest integer n such that $Ext_R^{n+1}(\bar{M},M)$= 0 for all flat module $\bar{M}$. The left cotorsion dimension of the R , Cot . D( R ) is the supremum of cotorsion dimension of R module. It is show in [ 5, corollary 7.26] that l
.cot. D( R) = sup{ qP / $\bar{M}$- flat}.

Proposition 1.6 :- Let M be an R-module and consider the following condition-

1) M is a quasi –w projective module
2) $Ext_R^i(M,P)$= 0 for all i >0 and the projective modules P.
3) $Ext_R^i(M,P)$= 0 for all i >0 and all module P with finite projective dimension.

$Proof$ :- (1)→(2) It is trivial
(2)↔(3) Result by dimension shifting.

Let $\bar{M}$ be a flat module. By Lazard's Theorem [ 2 section 1N.6 The. 1] there is a direct system $(L_i)i \in I$ of finitely generated free R-modules such that $\underset{\longrightarrow}{lim} L_i \cong \bar{M}$

If M is finitely presented from [2, Exercise 3, P-187] we have $Ext_R^i(M,\bar{M}) \cong \underset{\longrightarrow}{\lim} Ext_R^i(M,L_i)$. Thus in this case the implication (3) →(1) holds.

If l .cot . D (R) < ∞ then $qpd_R(\bar{M}) < \infty$. Hence, in this case also the implication (3)→(1) holds.

Proposition 1.7 :- The following statements are equivalent-

1) R is left perfect
2) Every flat module is quasi w-projective.

In particular if the class of all quasi-w projective modules are closed under direct limits then R is left Perfect

Proof :- If R is left perfect it is clear then every flat module is quasi-w projective. As t o the converse let $\bar{M}$ be a flat module. By (1) it is quasi-w projective and so projective by proposition 2.4. then R is left perfect. If the class of all quasi –w projective module is closed under direct limits then any direct limit of projective modules is both flat and quasi-w projective since every projective module is both flat and quasi-w-projective then by corollary 1.5 every direct limit of projective modules is projective. Thus R is left perfect.

Proposition 1.8 :- The following are equivalent –

(1) Every R-module is quasi-w projective
(2) R is quasi- Frobenius

Proof :- This follows from the fact that a ring is quasi-Frobenius if and only if every projective module is injective and that is quasi- Frobenius rings are Perfect.

A left (right) R-module M is said FP- injective if $Ext_R^1$ ( M,M) = 0 for every finitely presented left (right) R-module M.

A ring R is said to be FC if it is left and right coherent and left right self FP- injective.

Proposition 1.9 :- The following are equivalent-

1) R is FC
2) Every finitely presented (left and right ) module is quasi w-projective.

Proof :- Let M be a finitely presented right or left module and $\bar{M}$ be a flat right or left module then $\bar{M}$ is FP-injective by [8 , lemma 4.1], So , $Ext_R^i(M,\bar{M})$ = 0 for i >0. Thus M is quasi w-projective.
As to converse for any finitely presented right or left module M we have $Ext_R^i(M,R)$= 0 for all i >0 by 2 thus R is self right and left FP- injective.

Proposition 1.10 :- For any R-module M and any positive integer n the following assertions are equivalent –

1) qwpdR(M)≤ n
2) $Ext_R^i(M,\bar{M})$ = 0 for all i>n and a;; R-module $\bar{M}$ with finite flat dimension.
3) $If\ 0 \to G_n \to G_{n-1} \to \ldots\ldots\ldots\ldots\ldots \to G_0 \to M \to 0$

Is an exact sequence of modules with $G_0 \ldots\ldots\ldots\ldots G_{n-1}$ are quasi-w projective modules then Gn is a quasi w-projective module.

Proof :- (1) ↔ The Proof of this equivalence is standard homological algebra.

(2)→(1) obvious

(1) →(2) Set P=qwpdR(M). By induction on $m=fd_R(\bar{M})$ we prove that $Ext_R^i(M,\bar{M})$ =0 for all i>P. The induction start is given by (1). If m>0 pick the short exact sequence $0 \to \bar{M} \to P \to \bar{M} \to 0$ where P is a projective module. Clearly fdR $(\bar{M}')$ =m-1. Thus $Ext_R^i(M,\bar{M})$ =0 for all i>n. from the long exact sequence………

$\to Ext_R^i(M,P) \to Ext_R^i(M,\bar{M}) \to Ext_R^{i+1}(M,\overline{M'}) \to \ldots\ldots\ldots\ldots\ldots$

It is clear that $Ext_R^i(M,\bar{M}) = 0$ for all i >n.

Proposition 1.11 :- For any R module M qwpdR (M)≤ $pd_R(M)$ with equally if $fd_R(M)$ is finite.

Proof :- The first inequality follows from the fact that every projective module is quasi-w projective. Now set $qwpd_R(M) = n < \infty$ and consider an n-step projective resolution of M as follows 0→M'→Pn-1→……………→P1→P0→M→0 where all Pi are projective. Clearly M' is quasi w-projective. If $fd_R(M) < \infty$ then $fd_R(M') < \infty$ and then it is projective by proposition 1.4. Hewnce $Pd_R(M) \leq n$, and so the equality holds.

[2] Quasi w- projective dimension of rings :

Definition 2.1 :- The left quasi –w projective dimension of a ring R , l .qwpd(R) is defined by setting l .qwpd(R)= sup { qwpd(M) / M is a left R-module.

Theorem 2.2 :- Let R be a ring and n be a positive integer then the following are equivalent –

(1) l.qwpd( R) $\leq n$.
(2) Qwpd(R/I) $\leq n$ For every left ideal I of R
(3) idR$(\bar{M}) \leq n$ for all projective module P.

Proof :- (1)→(2) and (3)→(4) are obvious.
(2)→(3) Let $\bar{M}$ be a flat module. Since qwpdR(R/I) $\leq n$. For every ideal I of R we have $Ext_R^i(R/I,\bar{M})$ = 0 for all i>n. Thus using the Baer Criterion $id_R(\bar{M}) \leq n$ .
(4)→(1) Let M be an arbitrary module. Since idR(P) $\leq n$ for each projective module P we have $Ext_R^i(M,P)$= 0 for all >n and projective module P. By dimension shifting we get that $Ext_R^i(M,P)$= 0 for all i>n and all module P with finite

projective dimension. l.cot. D(R ) $\leq$ sup.{$id_R$(M)/ P projective} $\leq n$. Thus given a flat module $\bar{M}$ we have $qpd_R(\bar{M}) < \infty$. Hence $Ext_R^i(M, \bar{M})$= 0 for all i>n. consequently $qqwpd_R$ (M) $\leq n$.

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