On The Numerical Solution of Picard Iteration Method for Fractional Integro - Differential Equation

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Abstract: In this paper, the concept of Successive Approximation method also called the Picard Iteration Method (PIM) for solving Fractional integro-differential equations is introduced. The fractional derivative is considered in the Caputo sense [6]. The proposed method reduces the equation into a standard integro integral equation of the second kind. Some test problems are considered to demonstrate the accuracy and the convergence of the presented method.

Numerical results show's that this approach is easy and accurate when applied to fractional integro-differential equations.

Keywords: Picard Iteration method; Volterra fractional integro-differential equation; Fredholm fractional integro-differential equation.

I. Introduction

Fractional calculus has a long history from 30 September 1695, when the derivative of order $\alpha = 1/2$ has been described by Leibniz [7]. The theory of derivatives and integrals of non-integer order goes back to Leibniz, Liouville, Grǜnwald, Letnikov and Riemann. There are many interesting books about fractional calculus and fractional differential equations [7]. Our main focus is the Caputo definition which turns out to be of great usefulness in real-life applications.

In the past decade, mathematicians have devoted effort to the study of explicit and numerical solutions to linear and nonlinear fraction differential equations [3-4]. An extensive amount of research has been done on fractional calculus, such as [5]. Over the past few years, a number of fractional calculus applications are being used and in the field of science, engineering and economics [6]. Research on linear and non-linear differential equations and linearization techniques has gained quite a momentum due the rapidly proliferating use and recent developments of fractional calculus in these fields [5].

II. Preliminaries

Definition 1

The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$ of the function $f \in C_\mu, \mu \geq -1$ is defined as

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\mu)^{\alpha-1} f(\mu) d\mu, \qquad (1)$$

$\alpha > 0, x > 0$

$$J^{\alpha} f(x) = f(x) \qquad (2)$$

When we formulate the model of real-world problems with fractional calculus, The Riemann–Liouville have certain disadvantages. Caputo proposed in his work on the theory of viscoelasticity [9] a modified fractional differential operator D_*^{α} .

Definition 2

The fraction derivative of $f(x)$ in Caputo sense is defined as

$$D^{\alpha} f(x) = J^{m-\alpha} D^{m} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-\eta)^{m-\alpha-1} f^{(m)}(\eta)\, d\eta$$

For (3)

$m - 1 < \alpha \leq m, m \in N, x > 0, f \in C_{-1}^{m}$

Definition 3

The left sided Riemann–Liouville fractional integral operator of the order $\mu^3 0$ of a function $f \in C_{\mu}$, $\mu^3 - 1$, is defined as

$$j^{\alpha} f(x) = \frac{1}{\Gamma(x)} \int_0^1 \frac{f(x)}{(x-t)^{1-\alpha}} dt, \ \alpha > 0, x > 0, \qquad (4)$$

$$j^{\alpha} f(x) = f(x) \qquad (5)$$

Definition 4

Let $\in C_{-1}^{m}$ $1, m \in N \cup \{0\}$. Then the Caputo fractional derivative of $f(x)$ is defined as

$$D^{\alpha} f(x) = \begin{cases} j^{m-\alpha} f^{m,(x)}, \ m-1 < \alpha \leq m, \ meN, \\ \frac{D^{m} f(x)}{Dx^{m}}, \alpha = m \end{cases} \qquad (6)$$

Hence, we have the following properties:

(1) $j^{\alpha} j^{v} f = j^{\alpha+v} f, \alpha, v > 0, f \in C_{\mu,} \mu > 0$

$j^{\alpha} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}, \alpha > 0, \gamma > -1, x > 0$

$j^{\alpha} D^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} f^{k}(0^{+}) \frac{x^{k}}{k}, \ x > 0, \ m-1 < \alpha \leq m$

$D j^{\alpha} f(x) = f(x), x > 0, m-1 < \alpha \leq m$

$DC = 0, C \text{ is the constant}$

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$$\begin{cases} 0, \ \beta \in N_0, \ \beta < [\alpha], \\ D^{\alpha} x^{\beta} = \frac{\tilde{A}(\beta+1)}{\tilde{A}(\beta-\alpha+1)X^{\beta-\alpha}}, \ \beta \in N_0, \ \beta \geq [\alpha] \end{cases}$$

Where [α] denoted the smallest integer greater than or equal to α and $N_0 = \{0,1,2, \dots\}$

Definition 5

An integral equation is an equation in which the unknown function $y(x)$ appears under an integral-signs (4) and (5).

A standard integral equation $y(x)$ is of the form:

$$y(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)y(t)dt \qquad (7)$$

Where $g(x)$ and $y(x)$ are the limits of integration, λ is a constant parameter, and $K(x,t)$ is a function of two variable x and t called the kernel or the nucleolus of the integral equation. The function $y(x)$ that will be determined appears under the integral sign and also appears inside and outside the integral sign as well. It is to be noted that the limits of integration $g(x)$ and $h(x)$ may be both variables, constants or mixed [8].

Definition 6

An integro-differential equation is an equation in which the unknown function $y(x)$ appears under an integral sign and contain ordinary derivatives $y^{(n)}(x)$ as well. A standard integro-differential equation is of the form:

$$y^{(n)}(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)y(t)dt \qquad (8)$$

Integral equations and integro-differential equations are classified into distinct types according to limits of integration and the kernel $K(x;t)$ are as prescribed before [7]

1. If the limits of the integration are fixed, then the integral equation is called a Fredholm integral equation and is of the form:
   $$y(x) = f(x) + \lambda \int_{a}^{b} K(x,t)y(t)dt \qquad (9)$$

2. If at least one limits is a variable, then the equation is called a Volterra integral equation and is given as:

   $$y(x) = f(x) + \lambda \int_{a}^{b} K(x,t)y(t)dt \qquad (10)$$

III. Fundamentals of The Picard Iteration Method

The successive approximations method, also called the Picard iteration method provides a scheme that can be used for solving initial value problems or integral equations. This method solves any problem by finding successive approximations to the solution by starting with an initial guess, called the zeroth approximation. As will be seen, the zeroth approximation is any selective real-valued function that will be used in a recurrence relation to determine the other approximations [8].

Given the linear Volterra integral equation of the second kind

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt, \qquad (11)$$

Where u(x) is the unknown function to be determined, $K(x,t)$ is the kernel, and λ is a parameter. The successive approximations method introduces the recurrence relation

$$u_n(x) = f(x) + \lambda \int_0^x K(x,t)\, u_{n-1}(t)dt,\ \ n \geq 1, \qquad (12)$$

Where the zeroth approximation $u_0(x)$ can be any selective real valued function. We always start with an initial guess for $u_0(x)$, mostly we select $0,1,x$ for $u_0(x)$.

In this paper, we will consider

$$u_0(x) = 0 \qquad (13)$$

and by using (12), several successive approximations $u_k, k \geq 1$ will be determined as

$$u_1(x) = f(x) + \lambda \int_0^x K(x,t)u_0(t)dt,$$

$$u_2(x) = f(x) + \lambda \int_0^x K(x,t)u_1(t)dt,$$

$$u_3(x) = f(x) + \lambda \int_0^x K(x,t)u_2(t)dt,$$

$$\vdots \qquad (14)$$

$$u_n(x) = f(x) + \lambda \int_0^x K(x,t)u_{n-1}(t)dt,$$

IV. Numerical Examples

Program 1:

Consider the following fractional integro-differential equation:

$$D^{\frac{3}{4}}(y(t)) = \frac{6t^{9/4}}{\Gamma\left(\frac{13}{14}\right)} + \left(\frac{-t^2 \exp(t)}{5}\right)y(t) + \int_0^t exp(t)xy(x)dx,\ \ 0 \leq x \leq 1, \qquad (15)$$

Subject to $y(0) = 0$ with exact solution $y(t) = t^3$ (16)

Applying $J^{3/4}$ to both sides of (15) and applying (6) yields

$$y(t) = \frac{6}{\Gamma\left(\frac{13}{14}\right)} J^{3/4}\left[t^{9/4}\right] - \frac{1}{5} J^{\frac{3}{4}}\left[\frac{t^2 \exp(t)}{5}\right] y(t) + J^{3/4}\left[\exp(t) \int_0^t xy(x)dx\right] \qquad (17)$$

Applying:

$y_0(t) = 0$ and (12), we have the following iterations:

$$y_1(t) = y_2(t) = y_3(t) = y_4(t) = y_5(t) = y_6(t) = t^3$$

Which coincides with the exact solution.

Problem 2:

Consider the following fractional integro-differential equation:

$$D^{\frac{1}{2}}(u(x)) = \frac{\left(\frac{8}{4}\right)x^{\frac{3}{2}} - 2x^{1/2}}{\sqrt{\pi}} + \frac{x}{12} + \int_0^1 xtu(t)dt, \quad 0 \le x \le 1, \qquad (18)$$

Subject to $u(0) = 0$ with exact solution $B(x) = x^2 - x$ (19)

Applying $J^{1/2}$ (Half fractional integral) to both sides of (18) and applying (6) yields

$$u(x) = x^2 - x + \frac{1}{2}J^{0.5}[x] + J^{0.5}\left[\int_0^1 xtu(t)dt\right] \qquad (20)$$

Choosing $u_0(x) = 0$ and applying (12), gives:

$$u_1(x) = x^2 - x + \frac{x^{3/2}}{9\sqrt{\pi}}$$
$$u_2(x) = x^2 - x + \frac{8x^{3/2}}{189\pi}$$
$$u_3(x) = x^2 + \frac{32x^{3/2}}{567\pi\sqrt{\pi}}$$
$$u_4(x) = x^2 - x + \frac{256x^{3/2}}{11907\pi^2}$$
$$u_5(x) = x^2 - x + \frac{2048x^{3/2}}{250047\pi^{5/2}}$$
$$u_6(x) = x^2 - x + \frac{16384x^{\frac{3}{2}}}{5250987\pi^3}$$

Therefore, the series solution is

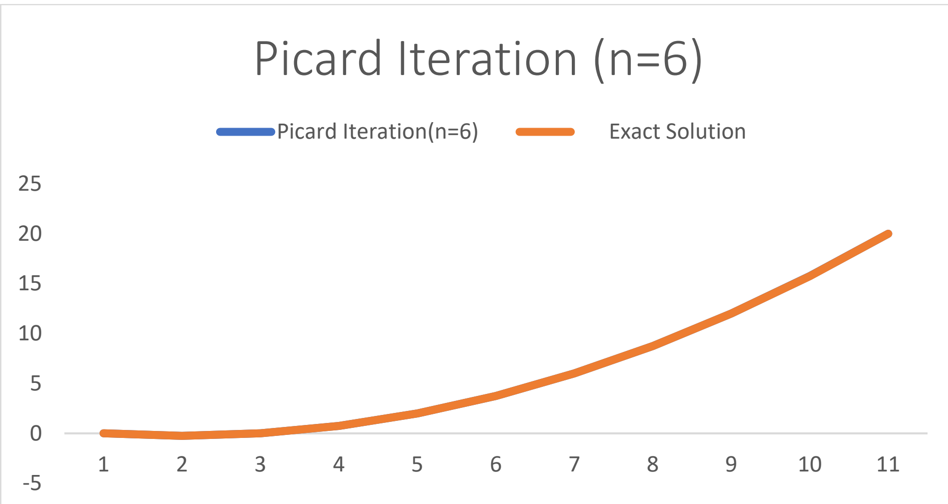
$$u(x) = x^2 - x + \frac{16384x^{\frac{3}{2}}}{5250987\pi^3}$$

Table of Numerical results for Problem 2

X	Picard(n=6)	Exact Solution	Abs. Error
0	0	0	0
0.5	-0.249964436	-0.25	3.55644E-05
1	0.000100591	0	0.000100591
1.5	0.750184798	0.75	0.000184798

2	2.000284515	2	0.000284515
2.5	3.750397622	3.75	0.000397622
3	6.000522688	6	0.000522688
3.5	8.750658662	8.75	0.000658662
4	12.00080473	12	0.00080473
4.5	15.75096024	15.75	0.000960239
5	20.00112465	20	0.001124645

Graphical Representation of Problem 2:



V. Conclusion

In this paper, Successive approximation method also known as Picard Iteration method (PIM) was applied to a transformed fractional integro-differential equations, the results when compared with other methods yields a better results.

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