The Application Of Adomian Decomposition Technique To Volterra Integral Type Of Equations

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Abstract- In this paper we introduced the Adomian Decomposition method. We explained the analytical points and then used the operator form obtained to solve problem of Volterra Integral type. The method is also illustrated by partial differential equation. The examples given are integral differential equations as well as partial differential equations.

Index Terms- Adomian method (ADM), decomposition, Volterra Integral, Differential Equations

I. Introduction

The Adomian decomposition method (ADM) is a semi analytical method for solving ordinary and partial nonlinear differential equations. The method was developed from 1970 to the 1990 by George Adomian the chair of the center for applied mathematics at the university of Georgia in (USA) it was further extended to (stochastic systems) by using the integral The aim of the method is to provide a unified theory for the solutions of partial differential equations (PDE) and this aim has been superseded by the more general theory of the homology analysis method. The crucial aspect of the method employment of the Adomian polynomial is that it allows for solution convergence of the nonlinear partial differential equation. Without simply linearizing the system. These polynomials mathematically generalize to a McLaren series about on arbitrary external parameter, which gives the solution method more flexibility than direct Taylor series expansion.

The Adomian Decomposition Method has been receiving much attention in recent years in applied mathematics in general, and in the area of series solutions in particular. The method proved to be powerful, effective and can easily handle a wide class of linear or nonlinear ordinary or partial differential equation, and linear or nonlinear integral equations. The decomposition method demonstrates fast convergence of the solution and therefore provides several significant advantages.

To show that the method is successfully used to handle most types of partial differential equation that appear in several physical models and scientific applications, the method attacks the problem in a direct way and straight forward fashion without using the linearization, perturbation or any other restrictive assumption that may change the physical behavior of the model under discussion .

II. The Analysis of Adomian Method:[6]

The Adomian decomposition method consists of decomposing the unknown function $u(x,y)$ of any equation in sum of an intuitive number of component.

It's defined by the decomposition series.

$$u(x,y)=\sum_{n=0}^{\infty}u_n\,(x,y), \qquad (1)$$

Where the component $u_n(x,y), n\geq 0$ are to be determined in
a recursive manner. The decomposition method concerns itself with finding the components $u_0,u_1,u_2,\ldots$ individually. As will be seen thorough recursive relation that usually involve simple integrals.

To give a clear an overview of Adomian Decomposition Method, we first consider the linear differential equation written in an operator from by

$$Lu+Ru=g \qquad (2)$$

Where L is operator, mostly, the lower order derivative which is assumed to be invertible,R is other linear differential operator. We next apply the inverse operator L^{-1} to both side of equation (2) we obtain

$$L^{-1}(Lu+Ru)=L^{-1}(g)$$
$$L^{-1}L(u)+L^{-1}(Ru)=L^{-1}(g)$$

(3)

$\Longrightarrow u+L^{-1}(Ru)=f \;$ where $f=L^{-1}(g)$

Then we get

$$u=f-L^{-1}(Ru) \qquad (4)$$

Where function f represents the terms arising from integrating.Adomian method defines the solution u by and infinite series of components given by:

$$u=\sum_{n=0}^{\infty}u_n \quad , \qquad (5)$$

Where the components u_0,u_1,u_2 are usually recurrently determined substituting it to both side of (4) and (5) it leads to

$$\sum_{n=0}^{\infty} u_n = f - L^{-1}\left(R\left(\sum_{n=0}^{\infty} u_n\right)\right) \qquad (6)$$

For simplicity equation (6) can be written as:

$$u_0 + u_1 + u_2 \ldots = f - L^{-1}(R(u_0 + u_1 + u_2 \ldots)) \qquad (7)$$

Now Respectively, We consider the system of nonlinear differential equations:

$$\frac{dy}{dt} = f(t, y), y \in \mathbb{R}^d, f: \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d \qquad (8)$$

With initial condition $y(0) = y_0 \in \mathbb{R}^d$ Assume that f is analytic near $y = y_0$ and $t = 0$. It is equivalent to solve the initial value problem for (8) and the Volterra integral equation

$$y(t) = y_0 + \int_0^t f(s, y(s))\, ds. \qquad (9)$$

To set up the Adomian method, consider y in the series form:

$$y = y_0 + \sum_{n=1}^{\infty} y_n, \qquad (10)$$

and write the nonlinear function $f(t, y)$ as the series of functions.

$$f(t, y) = \sum_{n=1}^{\infty} A_n(t, y_0, y_1, \ldots, y_n) \qquad (11)$$

The dependence of A_n on t and y_0, A_n is obtained by

$$A_n = \frac{1}{n!}\frac{d^n}{d\epsilon^n} f\left(t, \sum_{n=1}^{\infty} \epsilon^i y_i\right)\Bigg|_{\epsilon=0}, n = 0,1,2, \ldots \qquad (12)$$

So that, Substitution of (10) and (11) into (9) gives recursive equation for y_{n+1} in terms of $(y_0, y_1, y_2, \ldots, y_n)$

$$y_{n+1}(t) = \int_0^t A_n(s, y_0(s), y_1(s), \ldots, y_n(s))ds, \quad n = 0,1,2, \ldots \qquad (13)$$

Convergence of series (10) obtained by (13).

III. The Methodology

3.1 Methodology for Integral Equation:[8]

The Adomian decomposition method (ADM) was introduced and developed by George Adomian in early 1990. Denoted by expressing $u(x)$ in the form of a series:

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \quad n \geq 0 \qquad (14)$$

With $u_0(x)$ as the term outside the integral sing. The integral equation is:

$$u(x) = f(x) + \lambda \int_0^x k(x, t)\, u(t)\, dt \qquad (15)$$

and hence $u_0(x) = f(x)$
substituting equation (14) in to (15)

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_0^x k(x, t) \sum_{n=0}^{\infty} u_n(t)dt \qquad (16)$$

Or equivalently

$$u_0(x) + (x) + u_2(x) + \cdots = f(x) + \lambda \int_0^x k(x, t)\, [u_0(t) + u_1(t) + \cdots]dt \qquad (17)$$

So that

$$u_0(x) = f(x)$$

$$u_1(x) = \lambda \int_0^x k(x, t)\, u_0(t)\, dt$$

$$u_2(x) = \lambda \int_0^x k(x, t)\, u_1(t)\, dt$$

$$u_3(x) = \lambda \int_0^x k(x, t)\, u_2(t)\, dt$$

$$\vdots$$

$$u_{n+1}(x) = \lambda \int_0^x k(x, t)\, u_n(t)\, dt$$

Where, the components $u_0(x), u_1(x), u_2(x), u_3(x), \ldots$ are completely determined.

The kernel $k(x, t)$ and the function $u_n(t)$ are given real valued functions, and λ is parameter. Equation (15) is called Volterra of the second kind.

3.2 Methodology for Differential Equations: [4]

The Adomion decomposition method denoted by

$$u(x, y) = \sum_{n=0}^{\infty} u_n(x, y), \quad n \geq 0 \qquad (18)$$

We first consider the linear differential equation written in an operator form by

$$Lu + Ru = g \qquad (19)$$

Where L is operator, mostly, the lower order derivative which is assumed to be invertible, R is other linear differential operator and g is a source term. Now we apply the inverse operator L^{-1} to both sides of the equation (19) and using the given condition to obtain

$$u = F - L^{-1}(Ru)(20)$$

Where the function represents the terms arising from integrating the source term g and from using the given conditions that are assumed to be prescribe Adomion decomposition method defines the solution u by an in finite series of components given by

$$u = \sum_{n=0}^{\infty} u_n \qquad (21)$$

Substituting (21) into both sides of (20)

$$\sum_{n=0}^{\infty} u_n = F - L^{-1}\left(R\left(\sum_{n=0}^{\infty} u_n\right)\right) \qquad (22)$$

Or equivalently

$$u_0 + u_1 + u_2 + u_3 + \cdots = F - L^{-1}[R(u_0 + u_1 + u_2 + u_3 + \cdots)] \qquad (23)$$

$$u_0 = F$$

$$u_{k+1} = -L^{-1}(R(u_k)) \ , \quad k \geq 0 \qquad (24)$$

Or equivalently

$u_0 = F$

$u_1 = -L^{-1}(R(u_0))$

$u_2 = -L^{-1}(R(u_1))$

$u_3 = -L^{-1}(R(u_2))$

$\vdots$

So that

$u(x, y) = u_1 + u_2 + u_3 + \cdots$

IV. Some Examples:

Example (1):

We shall use the Adomian decomposition method to solve the following Volterra integral equation:

$$u(x) = x^2 + \int_0^x (x - t)u(t)\,dt \qquad (25)$$

to do that let

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_0^x k(x,t) \sum_{n=0}^{\infty} u_n(t)dt \ , \quad n \geq 0 \qquad \to (*)$$

We notice that $f(x) = x^2, \lambda = 1, K(x,t) = (x - t)$. so that

$$u_0(x) = F(x) = x^2$$

$$u_1(x) = \lambda \int_0^x k(x,t)\, u_0(t)\, dt = \int_0^x (x - t)\, t^2 dt = \frac{x^4}{12}$$

$$u_2(x) = \lambda \int_0^x k(x,t)\, u_1(t)\, dt = \frac{1}{12}\int_0^x (x - t)t^4\, dt = \frac{x^6}{360}$$

$$u_3(x) = \lambda \int_0^x k(x,t)\, u_2(t)\, dt = \frac{1}{360}\int_0^x \frac{t^2}{2!}\,(x - t)\, t^6 dt = \frac{x^8}{20160}$$

and so on using above (*)gives the series solution:

$$u(x) = x^2 + x + \frac{1}{12}x^4 + \frac{1}{360}x^6 + \frac{1}{20160}x^8 + \cdots, \qquad (26)$$

that converges to the closed from solution:

$$u(x) = 2\cosh x \ - 2 \qquad (27)$$

Example (2):

Now we apply the Adomian decomposition method for the solution of the following Volterra integral equation:

$$u(x) = x - \frac{2}{3}\,x^3 - 2\int_0^x u(t)dt. \qquad (28)$$

Consider the series

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_0^x k(x,t) \sum_{n=0}^{\infty} u_n(t)dt \ , \quad n \geq 0 \qquad \to (*)$$

We notice that $f(x) = x - \frac{2}{3}\,x^3,\ \lambda = -2,\ K(x,t) = 1$

so that

$$u_0(x) = F(x) = x - \frac{2}{3}\,x^3$$

$$u_1(x) = \lambda \int_0^x k(x,t)\, u_0(t)\, dt = -2\int_0^x \left(t - \frac{2}{3}\,t^3\right) dt = -x^2 + \frac{x^4}{3}$$

$$u_2(x) = \lambda \int_0^x k(x,t)\, u_1(t)\, dt$$

$$= -2\int_0^x \left(-t^2 + \frac{t^4}{3}\right) dt = \frac{2}{3}\,x^3 - \frac{2x^5}{15}$$

$$u_3(x) = \lambda \int_0^x k(x,t)\, u_2(t)\, dt$$

$$= -2\int_0^x \left(\frac{2}{3}\,t^3 - \frac{2t^5}{15}\right) dt = -\frac{1}{3}\,x^4 + \frac{2x^6}{45}$$

and so on, using above (*) gives the series solution:

$$u(x) = x - \frac{2}{3}\,x^3 - x^2 + \frac{x^4}{3} + \frac{2}{3}\,x^3 - \frac{2x^5}{15} - \frac{1}{3}\,x^4 + \frac{2x^6}{45} + \cdots$$

hence

$$u(x) = x - x^2$$

Example (3):

In this example we apply Adomian decomposition for the following homogeneous partial differential equation:

$$xu_x + u_y = 3u \ , \qquad u(x,0) = x^2 \, , \ u(0,y) = 0 \qquad (29)$$

In an operator form, equation (29) becomes.

$$L_y u(x,y) = 3u(x,y) - xL_x u(x,y) \qquad (30)$$

Appling the inverse operator L_y^{-1} to both sides of (30) and using the given condition $u(x,0) = x^2$ yields

$$u(x,y) = x^2 + L_y^{-1}(3u - xL_x u) \qquad (31)$$

Substituting $u(x,y) = \sum_{n=0}^{\infty} u_n(x,y)$ in to both sides of (31) gives

$$\sum_{n=0}^{\infty} u_n(x,y) = x^2 + L_y^{-1}(3\left(\sum_{n=0}^{\infty} u_n(x,y)\right) - xL_x\left(\sum_{n=0}^{\infty} u_n(x,y)\right) \qquad (32)$$

By considering few terms of the decomposition of $u_n(x,y)$ equation (32) becomes.

$$u_0 + u_1 + u_2 + \cdots = x^2 + L_y^{-1} 3(u_0 + u_1 + u_2 + \cdots) - xL_x(u_0 + u_1 + u_2 + \cdots) \quad (33)$$

$u_0(x) = x^2$

$$u_{k+1}(x,y) = L_y^{-1}(3u_k - xL_x u_k), \quad k \geq 0 \quad (34)$$

The components $u_0, u_1, u_2, \ldots$ are thus determined as follows:

$u_0(x,y) = x^2$

$$\left.\begin{aligned} u_1(x,y) &= L_y^{-1}(3u_0 - xL_x u_0) = x^2 y \\ u_2(x,y) &= L_y^{-1}(3u_1 - xL_x u_1) = \frac{x^2y^2}{2!} \\ u_3(x,y) &= L_y^{-1}(3u_2 - xL_x u_2) = \frac{x^2y^3}{3!} \end{aligned}\right\}$$

Consequently, the solution is given by

$$u(x,y) = u_0 + u_1 + u_2 + \cdots = x^2\left(1 + y + \frac{y^2}{2!} + \cdots\right) = x^2 e^y$$

Example (4):

We shall repeat the Adomian decomposition method for the following homogeneous partial differential equation:

$$u_x - yu = 0 \ , \ u(0,y) = 1 \quad (36)$$

In an operator form, equation (36)

$L_x u(x,y) = yu(x,y)$(37)

Where the operator L_x is defined as

$$L_x = \frac{\partial}{\partial x} \quad (38)$$

Applying the integral operator L_x^{-1} to both sides of (37) and using the given condition that $u(0,y) = 1$ gives

$$u(x,y) = 1 + L_x^{-1}(y\, u(x,y) \quad (39)$$

Following the discussion presented above, we define the unknown function $u(x,y)$ by the decomposition series:

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y) \quad (40)$$

Inserting (40) in to both sides of (39) gives:

$$\sum u_n(x,y) = 1 + L_x^{-1}\left(y\sum_{n=0}^{\infty} u_n(x,y)\right) \quad (41)$$

Or equivalently

$$u_0 + u_1 + u_2 + \cdots = 1 + L_x^{-1}\big(y(u_0 + u_1 + u_2 + \cdots)\big) \quad (42)$$

$u_0(x,y) = 1$

$u_1(x,y) = L_x^{-1}(yu_0) = xy$

$u_2(x,y) = L_x^{-1}(yu_1) = \frac{1}{2!}x^2y$

$u_3(x,y) = L_x^{-1}(yu_2) = \frac{1}{3!}x^3y^3$

So that:

$$u(x,y) = u_0(x,y) + u_1(x,y) + u_2(x,y) + u_3(x,y) + \cdots = 1 + xy + \frac{1}{2!}x^2y^2 + \frac{1}{3!}x^3y^3 + \cdots \quad (35)$$

and in a closed form

$$u(x,y) = e^{xy}$$

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