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A Comparison of Chebyshev polynomials and Legendre polynomials in order to solving Fredholm integral equations

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Abstract- In this research we use the numerical solution method that is based on Chebyshev polynomials and Legendre polynomials, to solve non-singular integral equation, it is known as Fredholm integral equation of the second kind. We use these expansions because of their convergence and recurrence properties. Also both of them can be represented as trigonometric function on [1, -1]. First, we expand the unknown function in the integral equation based on the related formulas, then develop kernel of integral equation. To find these, we should try to find a function which can be represented as the solution of linear differential equation. Then substitution into the integral equation, we find the coefficients of the function. At the end of research the method will be illustrated by the mean of an example.

Mathematics Subject Classification- 45B05, 41A10, 42C05, 42C10

Index Terms- Chebyshev approximation, Fredholm integral equation, Legendre series approximation, the numerical solutions of integral equations.

I. Introduction

The name integral equation for any equation involving the unknown function $f(x)$ under the integral sign was introduced by du Bois Reymond in 1888. However, the early history of integral equations goes back a considerable time before that to Laplace in 1782. Later Abel was led to an integral equation in connection with mechanical problem and obtained two solution of it; after this, Liouville investigated an integral equation which arose in the course of his researches on differential equation and discovered an important method for solving integral equations. In some problems mathematical representation appear directly in the form of the integral equations. Some problems have direct representation in term of differential equations with auxiliary conditions and may also be reduced to integral equations. Further information may be found in [1], [10], and [12]. Integral equations are one of the useful mathematical tools in applied analysis; here we introduce some applications of the integral equations such that, the problems of mechanical vibration, the problem of forecasting human population, determining the energy spectrum of neutrons, automatic control of rotating shaft, torsion of wire, and etc. But most of these equations are very difficult to solve. It is worth noting that Integral Equations often do not have an analytical solution, and must be solved numerically. There are solutions such Laplace transform, Fourier transform, and Mellin transform for some integral equations, but many of integral equations cannot be solved by these methods and should be solved by numerical methods (see [2]). The most common method of solution of integral equation is by the use of finite differences. In [3] Fox and Goodwin use the Gregory quadrature formula for the evaluation the integral equations. In this research we try to find the numerical solution of non-singular linear integral equations by the direct expansion of the unknown function, $f(x)$ into a series of Chebyshev polynomials of the first kind and into a series of Legendre polynomials (as discussed by Elliott [4]). Then we use given integral equation to obtaining coefficient. In [5] we see the properties of the Chebyshev polynomials together produce an approximating polynomial which minimizes error in its application. This is different from the least squares approximation where the sum of the squares of the errors is minimized; the maximum error itself can be quite large. In the Chebyshev approximation, the average error can be large but the maximum error is minimized. Chebyshev approximations of a function are sometimes said to be mini-max approximations of the function. Chebyshev polynomials form a special class of polynomials especially suited for approximating other functions. They are used in many areas of numerical analysis. It is assumed that expansions of given functions can be found and for functions whose expansions cannot be found in given manners, some curve fitting technique can be used. The Legendre polynomials [7] are one of the important sequences of orthogonal polynomials which has been extensively investigated and applied in interpolation and approximation theory, numerical integration, the solution of the second- and fourth-order elliptic equations, computational fluid dynamics, etc. It is not only powerful tool for the approximation of functions that are difficult to compute, but also essential ingredient of numerical integration and approximate solution of differential and integral equations. The Legendre spectral methods has excellent error properties in the approximation of a smooth function. The orthogonal polynomial expansion occurs in a wide range of practical problems and applications and plays an important role in many fields of mathematics and physics.

II. METHODS

Linear integral equations can be divided into two types depending upon the limits of the integral. An important integral equation of a general type is

$$f(x)=F(x)+\lambda\int_a^b K(x,y)\,f(y)\,dy\,,$$

where $F(x)$ is a given continuous function, λ is a parameter, a, b are finite constants, $K(x, y)$ is called the kernel and $f(x)$ is the unknown function. This integral equation is known as a Fredholm equation of the second kind. It was observed by Volterra that an equation of this type could be regarded as a limiting from of a system of linear equations. If F equals zero then we have homogeneous Fredholm equation of the second kind. When the upper limit of the integral is the variable x, the equation is known as a Volterra equation of the second kind.

A. Chebyshev polynomials method

Chebyshev[1] series based on the Chebyshev polynomials of the first kind are the most useful ones and have faster uniform convergence. For convenience, we write the Chebyshev series as

$$f(x)=\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}a_nT_n(x)\,. \tag{1}$$

Where

$$a_n=\frac{2}{\pi}\int_{-1}^{1}\frac{f(x)\;T_n(x)}{\sqrt{1-x^2}}\,dx=\frac{2}{\pi}\int_0^{\pi}f(\cos\;\theta)\;\cos(n\theta)\,d\theta,\;\;\theta=\cos^{-1}x\,. \tag{2}$$

All Chebyshev polynomials satisfy a three term recurrence relation,

$$2T_m(x)T_n(x)=T_{m+n}(x)+T_{m-n}(x)\,. \tag{3}$$

In order to solve Fredholm integral equation we need the integral of a product of two functions. First we must find the Chebyshev expansion of f(x).g(x). Suppose

$$f(x)=\tfrac{1}{2}a_0+\sum_{m=1}^{\infty}a_m\,T_m(x),\quad g(x)=\tfrac{1}{2}b_0+\sum_{n=1}^{\infty}b_n\,T_n(x)\,.$$

Let $h(x) = f(x).g(x)$, and $h(x)=\tfrac{1}{2}d_0+\sum_{n=1}^{\infty}d_nT_n(x)$, we get

$$h(x)=\tfrac{1}{4}a_0b_0+\tfrac{1}{2}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}a_mb_n\,T_{m+n}(x)+\tfrac{1}{2}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}a_mb_n\,T_{m-n}(x)$$

We know that d_0 is coefficient of $T_0(x)$. Hence

$$d_0=\tfrac{1}{2}a_0b_0+\sum_{m=1}^{\infty}a_mb_m\,. \tag{4}$$

Continue with $h(x)$ and let $n=j-m$ in the first series and let $n=j+m$ in the second series, we get

$$h(x)=\tfrac{1}{4}a_0b_0+\tfrac{1}{2}a_0\sum_{j=1}^{\infty}b_j\,T_j(x)+\tfrac{1}{2}\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}a_m[b_{|j-m|}+b_{j+m}]T_j(x).$$

According to equation (4), we can write

$$d_j=\tfrac{1}{2}[a_0b_j+\sum_{m=1}^{\infty}a_m(b_{|j-m|}+b_{j+m})],\;j\geq 0\,. \tag{5}$$

We want the expansion of $I(x)$, where

[1] Another transliteration of the name is Tchebichef.

$$I(x)=\int_{-1}^{x} f(t)\,dt. \tag{6}$$

If we set

$$f(x)=\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}a_nT_n(x),\ I(x)=\tfrac{1}{2}b_0+\sum_{n=1}^{\infty}b_nT_n(x)\ ,$$

and put in equation (6) we get

$$I(x)=\tfrac{1}{2}a_0+\tfrac{1}{2}a_0\,T_1+\tfrac{1}{2}\sum_{n=2}^{\infty}a_n\left[\frac{\cos(n+1)t}{n+1}-\frac{\cos(n-1)t}{n-1}\right]\Big|_{\pi}^{\cos^{-1}x}+\tfrac{1}{4}a_1\,T_2(x)-\tfrac{1}{4}a_1$$

$$=\tfrac{1}{2}a_0-\tfrac{1}{4}a_1-\tfrac{1}{2}\sum_{n=2}^{\infty}\frac{2(-1)^n}{n^2-1}a_n\ +\tfrac{1}{2}\sum_{n=1}^{\infty}\frac{a_{n-1}-a_{n+1}}{n}T_n(x).$$

Let us now compare this result with expansion of $I(x)$; we see that

$$b_0=a_0-\tfrac{1}{2}a_1-\sum_{n=2}^{\infty}\frac{2(-1)^n}{n^2-1}a_n,\ b_n=\frac{a_{n-1}-a_{n+1}}{2n},\ n\geq 1 \tag{7), (8}$$

For computing $I(1)$ set $x=1$, therefore

$$I(1)=a_0-2\sum_{n=1}^{\infty}\frac{a_{2n}}{4n^2-1}. \tag{9}$$

It should be noted that to use Chebyshev polynomials and Legendre polynomials we must change the range of the variable x from (a, b) to (-1, 1). So

$$I=\int_{-1}^{1} f(x)\,g(x)\,dx\ . \tag{10}$$

Defining $f(x)$ and $g(x)$ as formula (1), and using equations (5) and (9) we find

$$I=a_0\left(\frac{1}{2}b_0-\sum_{j=1}^{\infty}\frac{b_{2j}}{4j^2-1}\right)+\sum_{n=1}^{\infty}a_n\left(b_n-\sum_{j=1}^{\infty}\frac{b_{|n-2j|}+b_{n+2j}}{4j^2-1}\right). \tag{11}$$

B. Legendre polynomials method

The Legendre expansion of unknown function in the range $-1\leq x\leq 1$ defined by

$$f(x)=\sum_{n=0}^{\infty}c_n\,P_n(x), \tag{12}$$

and this expansion known as the Fourier-Legendre series, where

$$c_n=\frac{2n+1}{2}\int_{-1}^{1} f(x)\,P_n(x)\,dx. \tag{13}$$

Legendre polynomials satisfy certain recurrence relations.one of the most important relations is the relation known as Bonnet's recursion formula and defined by

$$(n+1)\,P_{n+1}(x)-(2n+1)x\,P_n(x)+n\,P_{n-1}(x)=0, \qquad n\geq 1. \tag{14}$$

In similar manner suppose that

$$f(x)=\sum_{n=0}^{\infty}c_n\,P_n(x),\quad g(x)=\sum_{m=0}^{\infty}d_m\,P_m(x),\quad I=\int_{-1}^{1} f(x)\,g(x).$$

By using this relation

$$\int_{-1}^{1} P_n(x)\,P_m(x)\,dx=\frac{2}{2n+1}\delta_{m,n},\ \delta_{m,n}=\begin{cases}0 & if\quad m\neq n\\ 1 & if\quad m=n\end{cases}\ (\textit{Kronecker delta}),$$

as a result

$$I=\int_{-1}^{1} f(x)\,g(x)\,dx=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_n d_m \int_{-1}^{1} P_n(x)\,P_m(x)\,dx=\sum_{n=0}^{\infty}\frac{2c_n d_n}{2n+1}\,. \tag{15}$$

C. Solving Fredholm integral equation of the second kind

We classify Fredholm integral equation according to the kernel in two type:

1. Fredholm integral equation with non-separable kernel
2. Fredholm integral equation with separable kernel.

In this research we focus on the first type because they are not usually solvable by routine methods.

C1. Fredholm integral equation of the second kind with non-separable kernel

In most problem where we need numerical method the kernel will be non-separable. Now suppose we have this Fredholm equation,

$$f(x)=F(x)+\lambda\int_{-1}^{1} K(x,y)\,f(y)\,dy,$$

where we have to find $f(x)$. We try to approximate the kernel by function with one independent variable, and choose some values for another variable, then use methods mentioned. We write $f(x)$ as before, in Chebyshev or Legendre expansion. So

$$f(x)\approx\frac{1}{2}a_0+\sum_{n=1}^{N} a_n\,T_n(x)\,,\ \text{(Chebyshev expansion)} \tag{16}$$

$$f(x)\approx a_0+\sum_{n=1}^{N} a_n\,P_n(x)\,.\ \text{(Legendre expansion)} \tag{17}$$

Where N is generally unknown and it may be given in the problem or it can be estimated from perhaps and some physical grounds. That mean we have $(N+1)$ unknown coefficients. In order to determine these constants, we write down the expansion of integral equation at each $(N+1)$ points of x_i, where $i=1,2,\ldots,N+1$. Then expand $F(x)$ in Chebyshev or Legendre polynomials, and write down for each $(N+1)$ values. Then we must find the kernel expansion .Suppose that

$$K(x_i,y)=\frac{h(x_i,y)}{g(x_i,y)},\ \ I(x_i)=\int_{-1}^{1} K(x_i,y)f(y)\,dy\,.$$

We expand $h(x_i,y)$ and $g(x_i,y)$ according to an arbitrary method have been told. Then for each value of x_i we compute the expansion for kernel. We obtain

$$f(x_i)=F(x_i)+\lambda I(x_i)\quad \text{for}\quad i=1,2,\ldots,N+1\,. \tag{18}$$

Finally we have a system of $(N+1)$ equations for the $(N+1)$ unknown coefficients, which can be solved.

C2. Fredholm integral equation of the second kind with separable kernel

When the kernel is separable we can write

$$K(x,y)=g(x)\,h(y)\,.$$

Hence Fredholm integral equation of the second kind takes the form

$$f(x)=F(x)+\lambda\,g(x)\int_{-1}^{1} h(y)\,f(y)\,dy\,.$$

In similar manner as has been said in the previous section, first expand $f(x)$ according to the formula (16) or (17) based on the desired method. Then expand the other function in the terms of Chebyshev polynomials or Legendre Polynomials. From equation (11) or equation (15) we can compute value of I. In this case we have one independent variable so abstaining I is very simpler than previous case. At the end of the solution we can obtain coefficients of the unknown function $f(x)$ by equating coefficients of polynomials of the same degree on each side of Fredholm integral equation.

III. EXAMPLE

We want to find $f(x)$ from solving

$$f(x)=1+\frac{1}{\pi}\int_{-1}^{1}\frac{1}{1+(x-y)^2}\,f(y)\,dy$$

Which the kernel $K(x,y)$ is not separable. We assume y as an independent value and try to approximate to the kernel by choosing some values of x. In both methods we approximate to the function by mean of sixth degree polynomial and fourth degree polynomial.

For the first case we know f (x) is an even function, hence based on (16) and (17) we have four unknown coefficients in both methods; so we choose these four values, $x_i = 0, 0.5, 0.8, 1$.

A. Chebyshev polynomials Approach

The kernel satisfies the equation given by

$$(1+x_i^2)K(x_i, y) - 2x_i yK(x_i, y) + y^2 K(x_i, y) = 1. \qquad (19)$$

Suppose that

$$K(x_i, y) = \tfrac{1}{2} b_0(x_i) + \sum_{n=1}^{\infty} b_n(x_i) T_n(y).$$

From equation (19) and using equation (3) we get

$$\tfrac{1}{2}(1+x_i^2) b_0(x_i) + (1+x_i^2)\sum_{n=1}^{\infty} b_n(x_i) T_n(y) - x_i b_0(x_i) T_1(y) - x_i \sum_{n=1}^{\infty} b_n(x_i) T_1(y) T_n(y)$$

$$+\tfrac{1}{4} b_0(x_i)(T_2(y)+1) + \tfrac{1}{2}\sum_{n=1}^{\infty} b_n(x_i)(T_2(y)+1) T_n(y) = 1.$$

Hence

$$(\tfrac{3}{4} + x_i^2) b_0(x_i) - x_i b_1(x_i) + \tfrac{1}{4} b_2(x_i) = 1, \qquad (20)$$

$$(\tfrac{7}{4} + x_i^2) b_1(x_i) - x_i b_0(x_i) - x_i b_2(x_i) + \tfrac{1}{4} b_3(x_i) = 0, \qquad (21)$$

$$(\tfrac{3}{2} + x_i^2) b_n(x_i) - x_i [b_{n+1}(x_i) + b_{n-1}(x_i)] + \tfrac{1}{4}[b_{n+2}(x_i) + b_{n-2}(x_i)] = 0. \quad n \ge 2 \qquad (22)$$

The computed coefficients by these equations (in Chebyshev expansion of the kernel) for each value of x_i are given in table 1.

Table 1: Coefficients of the kernel produced by Chebyshev expansion.

n	$b_n(0)$	$b_n(0.5)$	$b_n(0.8)$	$b_n(1)$
0	1.414214	1.361549	1.252701	1.137729
1	0	0.31920	0.42286	0.43457
2	- 0.24264	- 0.12703	- 0.00841	0.04965
3	0	- 0.08453	- 0.06081	- 0.03079
4	0.04163	- 0.00300	- 0.02218	- 0.01912
5	0	0.01245	- 0.00023	- 0.00449
6	- 0.00714	0.00385	0.00293	0.00037
7	0	- 0.00091	0.00116	0.00070
8	0.00123	- 0.00085	0.00004	0.00025
9	0	- 0.00009	- 0.00014	0.00003
10	- 0.00021	0.00011	- 0.00006	- 0.00002
11	0	0.00004	- 0.00001	- 0.00001
12	0.00004	- 0.00001	0.00001	0
13	0	- 0.00001	0	0
14	- 0.00001	0	0	0
15	0	0	0	0

Evaluate $I(x_i)$ for each value of x_i by means of equation (11), giving

$$I(x_i) = a_0\left(\frac{1}{2} b_0(x_i) - \sum_{j=1}^{7} \frac{b_{2j}(x_i)}{4j^2 - 1}\right) + \sum_{n=1}^{6} a_n(x_i)\left(b_n(x_i) - \sum_{j=1}^{10} \frac{b_{|n-2j|}(x_i) + b_{n+2j}(x_i)}{4j^2 - 1}\right).$$

So we obtain

$$\begin{cases} I(0) = 0.78540\,a_0 - 0.71238\,a_2 + 0.03686\,a - 0.04217 a_6 \\ I(0.5) = 0.72322\,a_0 - 0.57161\,a_2 - 0.04902\,a_4 - 0.02328\,a_6 \\ I(0.8) = 0.63055\,a_0 - 0.41763\,a_2 - 0.10331\,a_4 - 0.02458\,a_6 \\ I(1) = 0.55358\,a_0 - 0.32602\,a_2 - 0.11278\,a_4 - 0.02975\,a_6 \end{cases}$$

By computing and substituting obtained values into equation (18), we have

$$\begin{cases} 0.25000\,a_0 - 0.77324\,a_2 + 0.98827 a_4 - 0.98658\,a_6 = 1 \\ 0.26979\,a_0 - 0.31805\,a_2 - 0.48440\,a_4 - 1.00782\,a_6 = 1 \\ 0.29929\,a_0 + 0.41294\,a_2 - 0.81032\,a_4 - 0.74437 a_6 = 1 \\ 0.32379\,a_0 + 1.10378\;\; a_2 + 1.03590\,a_4 + 1.00947 a_6 = 1 \end{cases}$$

Solving this system of equations gives,

$$f(x) = 1.77443 - 0.14006\,T_2(x) + 0.00494\,T_4(x) + 0.00040\,T_6(x)\,.$$

For approximating to the function by mean of polynomial of degree 4 we choose $x_i = 0, 0.5, 1$. By computation we obtain

$$f(x) = 1.77503 - 0.13968\,T_2(x) + 0.00453\,T_4(x)\,.$$

B. Legendre polynomials Approach

Now we use Legendre polynomials. We can write from (12)

$$K(x_i, y) = \sum_{n=0}^{N} b_n(x_i) P_n(y),\; yK(x_i, y) = \sum_{n=0}^{N} c_n(x_i) P_n(y),\; y^2 K(x_i, y) = \sum_{n=0}^{N} d_n(x_i) P_n(y).$$

Substitution into equation (19) gives,

$$(1 + x_i^2)\, b_0(x_i) - 2x_i c_0(x_i) + d_0(x_i) = 1\,, \tag{23}$$

$$(1 + x_i^2)\, b_n(x_i) - 2x_i c_n(x_i) + d_n(x_i) = 0 \quad \text{if} \quad n \ge 1. \tag{24}$$

Using the formula (13) and using Bonnet's recursion formula (14), hence

$$c_n(x_i) = \frac{n+1}{2n+3} b_{n+1}(x_i) + \frac{n}{2n-1} b_{n-1}(x_i)\,, \tag{25}$$

$$d_n(x_i) = \frac{(n+1)(n+2)}{(2n+3)(2n+5)} b_{n+2}(x_i) + \frac{(n+1)^2}{(2n+1)(2n+3)} b_n(x_i) + \frac{n^2}{4n^2-1} b_n(x_i) + \frac{n(n+1)}{(2n-1)(2n-3)} b_{n-2}(x_i) \tag{26}$$

The results are given in table 2.

Table 2: Coefficients of the kernel produced by Legendre expansion.

n	$b_n(0)$	$b_n(0.5)$	$b_n(0.8)$	$b_n(1)$
0	0.785398	0.723221	0.630547	0.553574
1	0	0.368197	0.459305	0.453645
2	- 0.35398	- 0.16775	0.00513	0.08067
3	0	- 0.14610	- 0.09733	- 0.04544
4	0.08296	- 0.00911	- 0.04348	- 0.03538
5	0	0.02631	- 0.00168	- 0.00988
6	- 0.01722	0.00950	0.00647	0.00053

From (15) we have

$$I(x_i) \approx \sum_{n=0}^{6} \frac{2a_n b_n(x_i)}{2n+1}\,.$$

So we can find $I(x_i)$ for each value of x_i by using the coefficients in table 2.

$$\begin{cases} I(0) = 1.57080\,a_0 - 0.14159\,a_2 + 0.01844\,a_4 - 0.00265\,a_6 \\ I(0.5) = 1.44644\,a_0 - 0.06710\,a_2 - 0.00202\,a_4 + 0.00146\,a_6 \\ I(0.8) = 1.26103\,a_0 + 0.00205\,a_2 - 0.00966\,a_4 + 0.00099\,a_6 \\ I(1) = 1.10714\,a_0 + 0.03227 a_2 - 0.00786\,a_4 + 0.00008\,a_6 \end{cases}$$

Computing the required values and then substitution into equation (18) gives

$$\begin{cases} 0.50000\,a_0 - 0.45493\,a_2 + 0.36913\,a_4 - 0.31166\,a_6 &= 1 \\ 0.53958\,a_0 - 0.10364\,a_2 - 0.28842\,a_4 + 0.32278\,a_6 &= 1 \\ 0.59858\;a_0 + 0.45935\,a_2 - 0.22993\,a_a - 0.39211\,a_6 &= 1 \\ 0.64759\,a_0 + 0.98973\,a_2 + 1.00250\,a_4 + 0.99997\,a_6 &= 1 \end{cases}$$

Finally by solving these equations and using equation (26), we obtain

$$f(x) = 1.82077 - 0.19059\,P_2(x) + 0.00862\,P_4(x) + 0.00088\,P_6(x)\,.$$

And now by choosing $x_i = 0, 0.5, 1$ and using table 2 we find

$$f(x) = 1.82128 - 0.18970\,P_2(x) + 0.00828\,P_4(x)\,.$$

C. Comparing the accuracy of two approaches

A comparison of the results with Fox and Goodwin is given in table 3.

Table 3: Comparison of the accuracy of methods.

x	$f(x)$ Fox and Goodwin	$f(x)$ Chebyshev 4th degree	$f(x)$ Chebyshev 6th degree	$f(x)$ Legendre 4th degree	$f(x)$ Legendre 6th degree
0	1.9191	1.91924	1.91903	1.91924	1.91902
±0.25	1.8997	1.89966	1.89959	1.89965	1.89958
±0.5	1.8424	1.84261	1.84239	1.84260	1.84239
±0.75	1.7520	1.75318	1.75199	1.75317	1.75199
±1	1.6397	1.63988	1.63971	1.63986	1.63968

Fox and Goodwin presented their result to *4D* with an estimated maximum error of 1×10^{-4} due to round-off error. We see a considerable improvement in accuracy obtained with extra computation.

IV. CONCLUSION

We can solve Fredholm integral equation of the second kind with the numerical solution using Chebyshev polynomials. This method is useful because of Chebyshev polynomials properties. We can also use the numerical solution based on Legendre polynomials. But in both methods we have to normalized the range of the independent variable such that t to $-1 \le t \le 1$. When we apply these methods on an example we find:

- The recurrence relation between coefficients of series are more complicated for Legendre polynomials than Chebyshev polynomials.
- Computing $I(x_i)$ with Legendre method is very simpler than Chebyshev method and if this is the criterion for selection of solving method, it is recommended that Legendre is a better choice.
- The computing time saved in using Legendre expansion instead of Chebyshev expansion will be more than the computing time saved in the expansion of kernel.
- Due to the large number of calculations, the accuracy of the kernel coefficients are very sensitive to the round-off error in the Chebyshev method.
- The method of calculating the coefficients of the kernel by Chebyshev polynomials should continue until we reach zero coefficients according to the desired accuracy but in the method of Legendre polynomials, number of required coefficients of the kernel is equal to the degree of approximation.
- In both given method when the degree of approximation is unknown, we can start with a low N and increase it until the desired accuracy is reached.

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