The Quintic Trigonometric Bézier Curve with single Shape Parameter

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Abstract- In this paper, we have constructed a quintic trigonometric Bézier curve with single shape parameter. The shape of the curve can be adjusted as desired, by simply altering the value of shape parameter, without changing the control polygon. The quintic trigonometric Bézier curve can be made close to the cubic Bézier curve or closer to the given control polygon than the cubic Bézier curve. Approximation property has been discussed.

Index Terms- Trigonometric Bézier Basis Function, Trigonometric Bézier Curve, Shape Parameter, Bézier Curve, Approximation properties.

I. INTRODUCTION

THe construction of curves and surfaces is a key issue in computer aided geometric design (CAGD). The CAGD method arise from the need of the efficient computer representation of practical curves and surfaces, which is very broadly used in engineering design. In CAGD, the parametric curves is the combination of basis functions and control points. The parametric representation of curves and surfaces with shape parameters have received attention in recent years. In recent years, the trigonometric with shape parameters play a very important role in CAGD in the design of curves. Many works have been done with the help of trigonometric polynomial for the representation of the curves and surfaces, see [1], [2], [6], [8], [10]. Therefore, it is clear that Bézier curves, the quadratic and cubic Bézier curves, have very wide applications. In recent years, trigonometric polynomial curves like those of Bézier type are considerably in discussion. Han [3] discussed a class of quartic trigonometric polynomial curves with a shape parameter. Han [4] presented pieceswise quadratic trigonometric polynomial curves with C^1 continuity analogous to the quadratic B-spline curves. Han [5] introduced cubic trigonometric polynomial curves with a shape parameter. The cubic trigonometric Bézier curves with two shape parameter was presented by Han et al [7]. Liu, et al [9] presented a study on class of TC- Bézier curves with shape parameters.

The paper is organized as follows. In section 2, the basis functions of the quintic trigonometric Bézier curve with single shape parameter are established and the properties of the basis function has been described. In section 3, quintic trigonometric Bézier curves and their properties are discussed. In section 4, By using shape parameter, shape control of the curves is studied and explained by using figure. In section 5, the approximability of the quintic trigonometric Bézier curves and cubic Bézier curves corresponding to their control polygon are shown.

II. QUINTIC TRIGONOMETRIC BÉZIER BASIS FUNCTIONS

In this section, definition and some properties of quintic trigonometric Bézier basis functions are given:

Definition 2.1: For selected real values of λ, where $\lambda \in$ [−2, 1] the following four functions of t ($t \in$ [0, 1]) are defined as quintic trigonometric Bézier basis functions with single shape parameter λ:

$$\begin{cases} b_0(t) = \frac{1}{4}(1 - \sin\frac{\pi}{2}t)^3(1 - \lambda\sin\frac{\pi}{2}t)^2 \\ b_1(t) = \frac{1}{2}\left[1 - \frac{1}{2}(1 - \cos\frac{\pi}{2}t)^3(1 - \lambda\cos\frac{\pi}{2}t)^2\right] \\ b_2(t) = \frac{1}{2}\left[1 - \frac{1}{2}(1 - \sin\frac{\pi}{2}t)^3(1 - \lambda\sin\frac{\pi}{2}t)^2\right] \\ b_3(t) = \frac{1}{4}(1 - \cos\frac{\pi}{2}t)^3(1 - \lambda\cos\frac{\pi}{2}t)^2 \end{cases} \tag{2.1}$$

Proof: (a) For $t \in [0,1]$ and $\lambda \in [-2,1]$, then

$0 \le (1 - \sin\frac{\pi}{2}t)^3 \le 1,$

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$0 \le (1 - \lambda\sin\frac{\pi}{2}t)^2 \le 9,$

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It is obvious that $b_i(t) \ge 0, i = 0,1,2,3.$

(b)

$$\sum_{i=0}^{3} b_i(t) = \frac{1}{4}(1 - \sin\frac{\pi}{2}t)^3(1 - \lambda\sin\frac{\pi}{2}t)^2 + \frac{1}{2}\left[1 - \frac{1}{2}(1 - \cos\frac{\pi}{2}t)^3(1 - \lambda\cos\frac{\pi}{2}t)^2\right] + \frac{1}{2}\left[1 - \frac{1}{2}(1 - \sin\frac{\pi}{2}t)^3(1 - \lambda\sin\frac{\pi}{2}t)^2\right] + \frac{1}{4}(1 - \cos\frac{\pi}{2}t)^3(1 - \lambda\cos\frac{\pi}{2}t)^2 = 1.$$

The remaining cases follow obviously.

Fig.1 shows the curve of the quintic trigonometric basis functions for $\lambda = -2, \lambda = -1, \lambda = 0, \lambda = 1$ are shows (a), (b), (c) and (d) respectively.

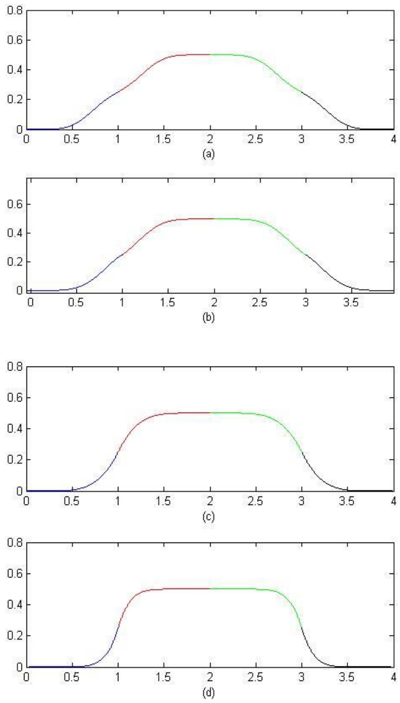

Figure 1: The quintic trigonometric basis functions for $\lambda=-2,\lambda=-1,\lambda=0,\lambda=1$

III. QUINTIC TRIGONOMETRIC BÉZIER CURVE

We construct the quintic trigonometric Bézier curve with shape parameter as follows:

Definition 3.1: Given the control points $P_i (i=0,1,2,3)$ in R^2 or R^3, then

$$r(t)=\sum_{i=0}^{3} P_i\, b_i(t) \qquad (3.1)$$

$t\in[0,1]$, $\lambda\in[-2,1]$ is called a quintic trigonometric Bézier curve with shape parameter.

From the definition of the basis functions some properties of the Quintic trigonometric Bézier curve can be obtained as follows:

Theorem 3.1: The Quintic trigonometric Bézier curve (3.1) have the following properties:

(a) Terminal properties:

$$r(0)=\frac{1}{4}[P_0+2P_1+P_2],$$
$$r(1)=\frac{1}{4}[P_3+2P_2+P_1] \qquad (3.2)$$
$$r'(0)=\frac{\pi}{8}(3+2\lambda)(P_2-P_0)$$
$$r'(1)=\frac{\pi}{8}(3+2\lambda)(P_3-P_1) \qquad (3.3)$$
$$r''(0)=\frac{\pi^2}{8}(\lambda^2+6\lambda+3)(P_0-P_2)$$
$$r''(1)=\frac{\pi^2}{8}(\lambda^2+6\lambda+3)(P_3-P_1) \qquad (3.4)$$

(b) Symmetry : P_0,P_1,P_2,P_3 and P_3,P_2,P_1,P_0 defined the same curve in different parametrizations, that is

$$r(t;\lambda;P_0,P_1,P_2,P_3)=r(1-t;\lambda;P_3,P_2,P_1,P_0) \qquad (3.5)$$
$t\in[0,1]$, $\lambda\in[-2,1]$.

(c) Geometric Invariance: The shape of the curve (3.1) is independent of the choice of coordinates, i.e., it satisfies the following two equations:

$$r(t;\lambda;P_0+q,P_1+q,P_2+q,P_3+q)=r(1-t;\lambda;P_0,P_1,P_2,P_3)+q$$

$$r(t;\lambda;P_0*T,P_1*T,P_2*T,P_3*T)=r(1-t;\lambda;P_0,P_1,P_2,P_3)*T \qquad (3.6)$$

where q is an arbitrary vector in R^2 or R^3 and *T* is an arbitrary $d\times d$ matrix, $d=2$ or 3.

(d) Convex hull property: From the non-negativity and partition of unity of basis functions, it follows that the whole curve is located in the convex hull generated by its control points.

IV. SHAPE CONTROL OF THE QUINTIC TRIGONOMETRIC BÉZIER CURVE

For $t\in[0,1]$, we rewrite (3.1) as follows:

$$r(t)=\sum_{i=0}^{3} P_i\, C_i(t)+\frac{1}{4}\lambda^2 sin^2\frac{\pi}{2}t(1-sin\frac{\pi}{2})^2(P_0-P_2)+\frac{1}{4}\lambda^2 cos^2\frac{\pi}{2}t(1-cos\frac{\pi}{2}t)^2(P_3-P_1)+\lambda sin\frac{\pi}{2}t(1-sin\frac{\pi}{2})^2\frac{(P_2-P_0)}{2}+\lambda cos\frac{\pi}{2}t(1-cos\frac{\pi}{2})^2\frac{(P_1-P_3)}{2} \qquad (4.1)$$

where

$$C_0(t)=\frac{1}{4}(1-sin\frac{\pi}{2}t)^3,$$
$$C_1(t)=\frac{1}{2}\left[1-\frac{1}{2}(1-cos\frac{\pi}{2}t)^3\right],$$
$$C_2(t)=\frac{1}{2}\left[1-\frac{1}{2}(1-sin\frac{\pi}{2}t)^3\right],$$
$$C_3(t)=\frac{1}{4}(1-cos\frac{\pi}{2}t)^3.$$

Obviously, shape parameter λ affect curves on the control edge $P_0-P_2, P_3-P_1, \frac{P_2-P_0}{2}, \frac{P_1-P_3}{2}$. Therefore as the shape parameter λ increases, the quintic trigonometric Bézier curve approximates the control polygon. The parameter λ controls the shape of the curve (4.1). In figure 2, The quintic trigonometric Bézier curve $r(t)$ gets closer to the control polygon as the value of the parameter λ increases. In figure 2, the curves are generated by setting the values of λ as $\lambda=-2$ (blue solid), $\lambda=-1$ (black dotted), $\lambda=0$ (green solid), $\lambda=1$ (red solid).

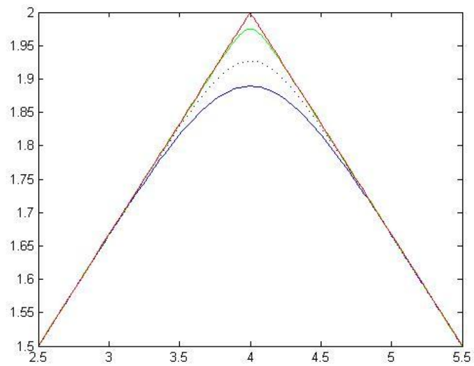

Figure 2: The effect on the shape of quintic trigonometric Bézier curve of altering the value of λ.

V. APPROXIMABILITY

Control polygons play an important role in geometric modeling. It is an advantage if the curve being modeled tends to preserve the shape of its control polygon. Now we show the relation of the quintic trigonometric Bézier curves and cubic Bézier curves with same control points.

Theorem 5.1: Suppose $P_0, P_1, P_2,$ *and* P_3 are not collinear; the relationship between quintic trigonometric Bézier curves $r(t)$ (3.1) and cubic Bézier curves $B(t) = \sum_{i=0}^{3} P_i \binom{3}{i} (1-t)^{3-i} t^i$ with the same control points $P_i (i = 0,1,2,3)$ are as follows:

$$r(0) = \frac{1}{4}[P_0 + 2P_1 + P_2], B(0) = P_0$$

$$r(1) = \frac{1}{4}[P_3 + 2P_2 + P_1], B(1) = P_3$$

$$r\left(\frac{1}{2}\right) - P^* = \frac{1}{2\sqrt{2}}\left(\sqrt{2}-1\right)^3\left(\sqrt{2}-\lambda\right)^2\left(B\left(\frac{1}{2}\right) - P^*\right) \quad (5.1)$$

where $P^* = \frac{1}{2}(P_1 + P_2)$

Proof: According to (3.2), we have

$$r(0) = \frac{1}{4}[P_0 + 2P_1 + P_2],$$

and

$$r(1) = \frac{1}{4}[P_3 + 2P_2 + P_1]$$

By simple computations,
$B(0) = P_0$ and $B(1) = P_3$
since

$$B(t) = (1-t)^3 P_0 + 3(1-t)^2 t P_1 + 3(1-t)t^2 P_2 + t^3 P_3$$

$$B\left(\frac{1}{2}\right) - P^* = \frac{1}{8}(P_0 - P_1 - P_2 + P_3) \quad (5.2)$$

and according to (5.2), we have

$$r\left(\frac{1}{2}\right) - P^* = \frac{1}{16\sqrt{2}}\left(\sqrt{2}-1\right)^3\left(\sqrt{2}-\lambda\right)^2 (P_0 - P_1 - P_2 + P_3)$$

$$r\left(\frac{1}{2}\right) - P^* = \frac{1}{16\sqrt{2}}\left(\sqrt{2}-1\right)^3\left(\sqrt{2}-\lambda\right)^2 \left(B\left(\frac{1}{2}\right) - P^*\right)$$

Then (5.1) holds.

Fig.3 shows the relationship between the quintic trigonometric Bézier curves curve and cubic Bézier curves. The quintic trigonometric Bézier curves is closer to cubic Bézier curve. The quintic trigonometric Bézier curve (blue solid) is closer to control polygon than the cubic Bézier curve (red solid) for all values of $\lambda \in [-2,1]$

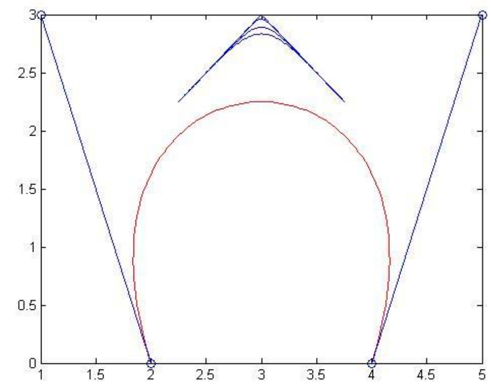

Figure 3: The relationship between the quintic trigonometric Bézier curve and cubic Bézier curve.

VI. CONCLUSION

In this paper, we have presented the quintic trigonometric Bézier curve with single shape parameter and analysis of quintic trigonometric Bézier curve are similar to the ordinary cubic Bézier curve. Each section of the curve only refers to the four control points and quintic trigonometric Bézier curve is closer to the cubic Bézier curve.

REFERENCES

[1] Dube,M., Sharma,R., Quadratic NUAT B-spline curve with multiple shape parameters, International Journal of Machine Intelligence, l3,(2011), 18-24.

[2] Dube,M., Sharma,R., Piecewise quartic trigonometric polynomial Bspline curve with two shape parameters, International Journal of Image and Graphics, (2012).

[3] Han,X., A class of general quartic spline curves with shape parameters. Computer Aided Geometric Design, 28, (2011), 151- 63.

[4] Han,X.,Piecewise quadratic trigonometric polynomial curves. Mathematics of Computation, 72, (2003), 1369-1378.

[5] Han,X.,The cubic trigonometric polynomial curve with a shape parameter. Computer Aided Geometric Design. 21 (2004) 535-548.

[6] Han, X., Quadratic trigonometric polynomial curves with shape parameter. Computer Aided Geometric Design, 19(7),(2002),479-502.

[7] Han, X., Ma,Y., Huang,X.,The cubic trigonometric Bézier curve with two shape parameters. Applied Mathematics Letters, 22,(2009), 226-231.

[8] Han,X., Ma,Y., Huang,L,X., Shape analysis of cubic trigonometric Bézier curve with a shape parameter. Applied Mathematics an Computational, 217, (2010), 2527-2533.

[9] Liu, H., Li, L., Zhang, D., Study on a class of TC-Bezier curve with shape parameters, Journal of Information and Computational Science, 8, (2011), 1217-1223.

[10] Wu,X., Han, X., Luo,S., Quadratic trigonometric spline curves with multiple shape parameters. In IEEE, International Conference onComputer-Aided Geometric Design and Computer Graphics,(2007), 413-6.

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