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Exponential – Log Logistic Additive Failure Rate Model

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Abstract- A combination of exponential and log – logistic failure rate model is considered and named it as exponential log-logistic additive failure rate model. An attempt is made to present the distributional properties, estimation of parameters, testing of hypothesis and the power of likelihood ratio criterion about the proposed model.

I. Introduction

It is well-known that in the theory of distributions, Normal distribution and exponential distributions are the basic models exemplified in theory of distributions. Specifically, exponential distribution is an invariable example for a number of theoretical concepts in reliability studies. It is characterized as CFR model also. In case of necessity for an IFR model, the choice falls on Weibull model with shape parameter more than 1 (>1), in particular taken as 2. Similar in shape, with common characteristics of Weibull, we have log-logistic distribution as another model. Log-logistic distribution has its own prominence as a life testing model. A log-logistic distribution is an IFR model and it is also a weighted exponential distribution. In this paper, we propose to combine an exponential model (CFR) and a log-logistic model (IFR) through their hazard functions to get two component series system reliability.

Log-logistic distribution is the model of transformed well-known logistic variate (Bain, 1974). Ragab and Green (1984) have studied the properties of log-logistic distribution and constructed its order statistics. Kantam *et al.* (2001&2006) constructed the acceptance sampling plans for log-logistic distribution. Kantam and Srinivasa Rao (2002) studied the modified maximum likelihood estimation in log-logistic distribution. Rosaiah *et al.* (2007) have constructed the confidence intervals for log-logistic distribution. Srinivasa Rao *et al.* (2013a) studied the properties, estimation and testing of linear failure rate model with exponential and half – logistic distribution. Srinivasa Rao *et al.* (2013b) have discussed the distributional properties, estimation of parameters and testing of hypothesis for additive failure rate model combining exponential and gamma distributions.

Because, exponential model and log-logistic model and the related works are not published in the available literature, we made an attempt to consider such a model for our study. In reliability studies, combinations of components forming series, parallel, k out of 'n' systems are quite popular. The survival probabilities of such systems are evaluated either by the systems as a whole or through the survival probabilities of the component that define the system. It is well known that in a series system of a finite number of components with independent life time random variables, the system reliability is equal to the product of the component reliabilities. If f(x), F(x), h(x) respectively indicates the failure density, failure probability, failure rate of a component with life time random variable 'X', then we know that

The reliability R(x) = 1-F(x)

$$R(x) = e^{-\int_0^x h(x)dx}$$

If a series system has two components with independent but non-identical life patterns explained by two distinct random variables say X_1, X_2, with respective failure densities, failure probabilities, failure rates as $f_1(x)$, $f_2(x)$; $F_1(x)$, $F_2(x)$; $h_1(x)$, $h_2(x)$, then the system reliability is given by

$$R(x) = e^{-\int_0^x [h_1(x)+h_2(x)]dx} \tag{1}$$

From the above equation, we can get the failure density and failure rate of the series system whose reliability is given by (1). Specifically, we consider a series system of two components with their respective life times modeled by exponential and log-logistic distribution.

The hazard functions of the exponential distribution with parameter 'λ' and a log- logistic distribution with parameters α, β are given by,

$$h_1(x) = \lambda,\ x > 0$$

and

$$h_2(x) = \frac{\left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\beta-1}}{1+\left(\frac{x}{\alpha}\right)^{\beta}},\ x > 0,\ \alpha, \beta > 0$$

The corresponding reliability functions are

$$e^{-\lambda x}\ and\ \left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-1}$$

The reliability of the series system is

$$R(x) = e^{-\int_0^x \left(\lambda + \frac{\left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\beta-1}}{1+\left(\frac{x}{\alpha}\right)^{\beta}} \right) dx} \qquad (2)$$

We consider the failure density corresponding to (2) as our exponential log-logistic additive failure rate model (ELLAFRM). The distributional properties, graphical natures for different choices of λ, β (assuming α = 1) are discussed in Section II. Estimation of parameters is presented in Section III. Likelihood ratio criterion and power of likelihood ratio criterion are given in Sections IV and V respectively. Summary and conclusions are given in Section VI.

II. Distributional Properties

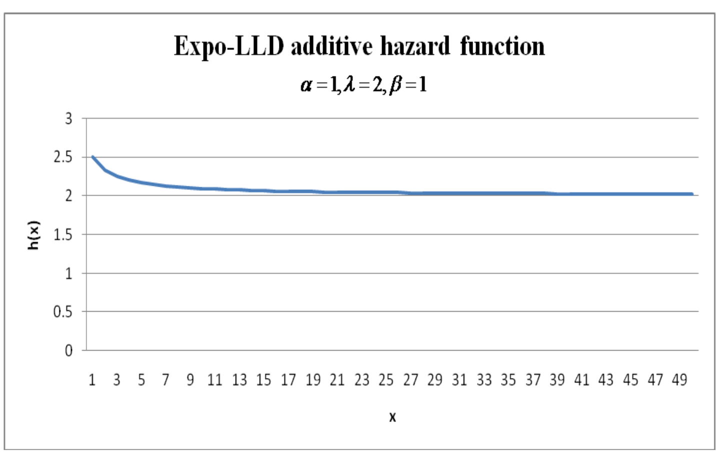
The probability density function (pdf), the cumulative distribution function (cdf), failure rate of ELLAFRM are respectively given by

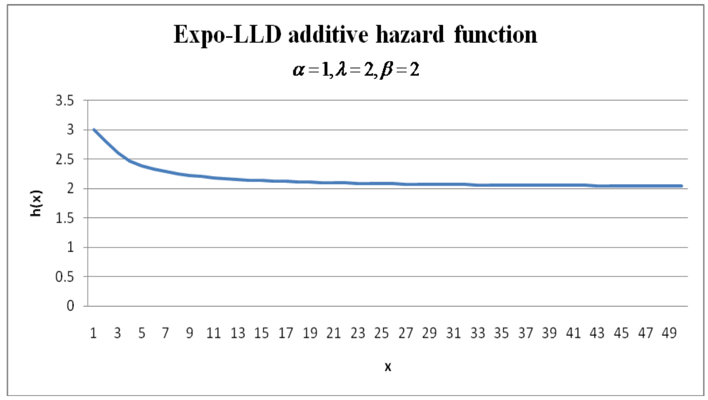
$$f(x) = e^{-\lambda x - \log\left[1+\left(x/\alpha\right)^{\beta}\right]} \left[\lambda + \frac{\beta x^{\beta-1}}{\alpha^{\beta} + x^{\beta}} \right], \qquad x>0, \alpha>0, \beta>0, \lambda>0 \qquad (3)$$

$$F(x) = 1 - e^{-\lambda x - \log\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]}, \qquad x>0, \lambda>0, \alpha>0, \beta>0 \qquad (4)$$

$$h(x) = \lambda + \frac{\beta x^{\beta-1}}{\alpha^{\beta} + x^{\beta}}, \quad x>0\ ;\ \lambda, \alpha, \beta > 0 \qquad (5)$$

The following graphs show that the hazard function is decreasing for initial values of x and then constant.





III. Maximum Likelihood Estimation

Let x_1, x_2, ……., x_n is a random sample of size ‘n’ drawn from the ELLAFRM with pdf given in equation (3), then the likelihood function is given by

$$L = \prod_{i=1}^{n} f(x_i; \lambda, \alpha, \beta)$$

$$\Rightarrow L = \pi_{i=1}^{n} \left[e^{-\lambda x_i - \log\left[1+\left(x_i/\alpha\right)^{\beta}\right]} \left[\lambda + \frac{\beta x_i^{\beta-1}}{\alpha^{\beta} + x_i^{\beta}} \right] \right] \qquad (6)$$

$$\Rightarrow \log L = \sum_{i=1}^{n} \log \left[e^{-\lambda x_i - \log\left[1+\left(x_i/\alpha\right)^{\beta}\right]} \left[\lambda + \frac{\beta x_i^{\beta-1}}{\alpha^{\beta} + x_i^{\beta}} \right] \right] \qquad (7)$$

The MLE’s of λ, α can be obtained with known β by solving the following likelihood equations

$$\frac{\partial \log L}{\partial \lambda} = 0 \qquad \Rightarrow \sum_{i=1}^{n} \left[-x_i + \frac{1}{\lambda + \frac{\beta x_i^{\beta-1}}{\alpha^{\beta} + x_i^{\beta}}} \right] = 0 \qquad (8)$$

$$\frac{\partial \log L}{\partial \alpha} = 0$$

$$\Rightarrow \beta \sum_{i=1}^{n} \left[\frac{x_i^{\beta}}{\alpha\left(\alpha^{\beta} + x_i^{\beta}\right)} - \frac{\beta x_i^{\beta-1} \alpha^{\beta-1}}{\left[\lambda\left(\alpha^{\beta} + x_i^{\beta}\right) + \beta x_i^{\beta-1}\right]\left[\alpha^{\beta} + x_i^{\beta}\right]} \right] = 0 \qquad (9)$$

Assuming that β is known, the equations (8) and (9) have to be solved simultaneously through iterative only with some well known numerical methods to get the MLEs of λ and α say $\hat{\lambda}$ and $\hat{\alpha}$ respectively .

However, by using a simple successive method, the ML equations (8) and (9) can be further simplified and get the following estimator (not ML estimator) for λ .

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i} \tag{10}$$

$$\mathrm{I}_{11} = -E\left(\frac{\partial^2 \log L}{\partial \lambda^2}\right) = -E\left[-\sum_{i=1}^{n} \frac{1}{\left[\lambda + \frac{\beta x_i^{\beta-1}}{\alpha^{\beta} + x_i^{\beta}}\right]^2}\right] \tag{11}$$

$$\mathrm{I}_{12} = \mathrm{I}_{21} = -E\left(\frac{\partial^2 \log L}{\partial \lambda \partial \alpha}\right) = -E\left[\sum_{i=1}^{n} \frac{\beta^2 x_i^{\beta-1} \alpha^{\beta-1}}{\left[\lambda\left(\alpha^{\beta} + x_i^{\beta}\right) + \beta x_i^{\beta-1}\right]^2}\right] \tag{12}$$

$$\mathrm{I}_{22} = -E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) = -E\left[\beta \sum_{i=1}^{n}\left[\frac{-x_i^{\beta}\left(\beta\alpha^{\beta-1}\alpha + \left(\alpha^{\beta} + x_i^{\beta}\right)\right)}{\left[\alpha\left(\alpha^{\beta} + x_i^{\beta}\right)\right]^2} - \beta x_i^{\beta-1}\left[\frac{(\beta-1)\alpha^{\beta-2}\left(\left[\lambda\left(\alpha^{\beta} + x_i^{\beta}\right) + \beta x_i^{\beta-1}\right]\left[\alpha^{\beta} + x_i^{\beta}\right]\right) - \left(\beta\alpha^{\beta-1}\left[\lambda\left(\alpha^{\beta} + x_i^{\beta}\right) + \beta x_i^{\beta-1}\right] + \lambda\beta\alpha^{\beta-1}\left(\alpha^{\beta} + x_i^{\beta}\right)\right)\alpha^{\beta-1}}{\left(\left[\lambda\left(\alpha^{\beta} + x_i^{\beta}\right) + \beta x_i^{\beta-1}\right]\left[\alpha^{\beta} + x_i^{\beta}\right]\right)^2}\right]\right]\right] \tag{13}$$

The estimated information matrix elements are

$$\hat{\mathrm{I}}_{11} = -E\left(\frac{\partial^2 \log L}{\partial \lambda^2}\right)\Big|_{\lambda=\hat{\lambda}}, \qquad \hat{\mathrm{I}}_{12} = \hat{\mathrm{I}}_{21} = -E\left(\frac{\partial^2 \log L}{\partial \lambda \partial \alpha}\right)\Big|_{|\lambda=\hat{\lambda},\, \alpha=\hat{\alpha}}$$

and $\hat{\mathrm{I}}_{22} = -E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right)\Big|_{\alpha=\hat{\alpha}}$

Accordingly, the exact variances of the MLEs are not mathematically tractable. However, the asymptotic variance, covariance of the estimates of the parameters is obtained using the following elements of the information matrix:

The estimated asymptotic dispersion matrix of the MLEs is given by the inverse of

$$\begin{bmatrix} \hat{\mathrm{I}}_{11} & \hat{\mathrm{I}}_{12} \\ \hat{\mathrm{I}}_{21} & \hat{\mathrm{I}}_{22} \end{bmatrix}$$

IV. Likelihood Ratio Type Criterion and Critical Values

Let us designate our distribution **ELLAFRM** as null population, say P_0. We call exponential distribution as alternate population, say P_1. We propose a null hypothesis H_0: "A given sample belongs to the population P_0" against an alternative hypothesis H_1: "The sample belongs to population P_1". Let L_1, L_0 respectively stand for the likelihood function of the sample with population P_1 and P_0. Both L_1 and L_0 contain the respective parameters of the population. The given sample is used to get the parameters of P_1, P_0, so that for the given sample, the value of L_1/L_0 is now estimated. If H_0 is true, L_1/L_0 must be small, therefore for accepting H_0 with a given degree of confidence, L_1/L_0 is compared with a critical value with the help of the percentiles in the sampling distribution of L_1/L_0.

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We have seen in Section 4, how to get the estimates of parameters. But the sampling distribution of L_1/L_0 is not analytical, we therefore resorted to the empirical sampling distribution through simulation. We have generated random samples of size 2(1)10 from the population P_0 with various parameter combinations and get the value of L_1, L_0 along with the estimates of respective parameters for each sample. The percentiles of L_1/L_0 at various probabilities are computed and are given in Tables 1 to 3.

Table -1: Percentiles of L_1/L_0 for $\alpha = 1,\ \lambda = 1,\ \beta = 2$

n	0.99	0.975	0.95	0.05	0.025	0.01
2	3.1748	1.8985	1.3072	0.4991	0.4938	0.4872
3	2.7711	1.7240	1.2677	0.3660	0.3547	0.3498
4	2.8102	1.6521	1.1175	0.2700	0.2625	0.2547
5	2.4595	1.8190	0.9772	0.2000	0.1924	0.1857
6	2.0317	1.3517	0.8563	0.1479	0.1422	0.1371
7	1.9982	1.1476	0.7408	0.1089	0.1402	0.1009
8	1.8364	1.0352	0.6722	0.0832	0.0772	0.0746
9	1.2912	0.8237	0.5504	0.0611	0.0575	0.0536
10	1.2227	0.7506	0.4759	0.0457	0.0425	0.0396

Table – 2: Percentiles of L_1/L_0 for $\alpha = 1,\ \lambda = 2,\ \beta = 2$

n	0.99	0.975	0.95	0.05	0.025	0.01
2	2.0360	1.5561	1.2253	0.7671	0.7583	0.7495
3	2.0246	1.5835	1.2806	0.6885	0.6795	0.6644
4	2.2464	1.6317	1.2722	0.6160	0.6031	0.5949
5	2.2765	1.7299	1.2907	0.5511	0.5395	0.5295
6	2.2025	1.6206	1.2341	0.4949	0.4853	0.4724
7	2.1648	1.5702	1.1961	0.4440	0.4316	0.4239
8	2.1820	1.6057	1.1646	0.4011	0.3876	0.3748
9	1.8864	1.4697	1.1364	0.3608	0.3485	0.3374
10	1.9067	1.3768	1.1642	0.3252	0.3138	0.2979

Table – 3: Percentiles of L_1/L_0 for $\alpha = 1,\ \lambda = 3,\ \beta = 2$

n	0.99	0.975	0.95	0.05	0.025	0.01
2	1.5899	1.3377	1.1367	0.8704	0.8651	0.8531
3	1.6767	1.3865	1.2034	0.8258	0.8191	0.8063
4	1.8069	1.4294	1.2104	0.7796	0.7700	0.7599
5	1.8615	1.5064	1.2212	0.7394	0.7266	0.7173
6	1.8182	1.4643	1.2159	0.6989	0.6908	0.6737
7	1.8471	1.4986	1.1990	0.6604	0.6515	0.6419
8	1.9015	1.5261	1.2276	0.6284	0.6155	0.5977
9	1.8025	1.4491	1.2234	0.5959	0.5860	0.5716
10	1.7324	1.4192	1.2444	0.5652	0.5522	0.5378

V. Power of Likelihood Ratio Criterion

In testing of hypothesis, we know that the power of a test statistic is the complementary probability of accepting a false hypothesis at a given level of significance. Let us conventionally fix 5% level of significance. So that the percentiles given in Table 1 to 3 under the column 0.05 shall become the critical values. We generate a random sample of sizes 2(1)10 from the population P_1 namely exponential. At this sample, we find the estimates of the parameters of P_1 and P_0 using the respective probability models. Accordingly, we get the estimates of L_1, L_0 for the sample from P_1.

Over repeated simulation runs, we get the proportion of values of L_1/L_0 that fall below the respective critical values of Table 1 to 3. These proportions would give the value of β, the probability of type II error. If the test statistic has a discriminating power, β must be small so that the power l-β must be large. Various power values are given in Table-4. We see that as 'n' increases, β is decreasing and hence l-β increases. We conclude that as long as 'n' increases the power of the likelihood ratio criterion increases. We therefore conclude that exponential can be a reasonable alternative to our model in small samples.

Table – 4

Power of Likelihood Ratio criterion for α = 1						
	$\lambda = 1,\ \beta = 2$		$\lambda = 2,\ \beta = 2$		$\lambda = 3,\ \beta = 2$	
n	***0.975***	***0.950***	***0.975***	***0.950***	***0.975***	***0.950***
2	0.883	0.826	0.783	0.747	0.729	0.691
3	0.860	0.817	0.734	0.687	0.654	0.624
4	0.854	0.797	0.713	0.658	0.607	0.576
5	0.882	0.789	0.716	0.631	0.569	0.519
6	0.837	0.769	0.674	0.604	0.530	0.470
7	0.824	0.755	0.634	0.569	0.495	0.431
8	0.815	0.744	0.626	0.550	0.452	0.390
9	0.795	0.725	0.589	0.529	0.407	0.370
10	0.797	0.718	0.562	0.514	0.369	0.332

VI. Summary and Conclusions

Exponential and log-logistic failure rate models are combined for the reliability studies and is named as Exponential – log logistic additive failure rate model. The distributional properties, estimation of parameters, testing of hypothesis and the power of likelihood ratio criterion about the proposed model are discussed.

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