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Zero –Free Regions for Analytic Functions

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Abstract- In this paper we find some interesting zero-free regions for a certain class of analytic functions by restricting the coefficients to certain conditions. Our results generalise a number of already known results in this direction.

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Index Terms- Zeros, Maximum Modulus, Analytic Function

I. Introduction and Statement of Results

Regarding the zero-free regions of analytic functions Aziz and Mohammad [1] have proved the following result:

Theorem A: Let $f(z)=\sum_{j=0}^{\infty} a_j z^j (\neq 0)$ be analytic for $|z|\le 1$ such that $a_j > 0$ and $a_{j-1} \ge t a_j,\ j=1,2,3,........,$ for some t>0 , then f(z) does not vanish in $|z|<t$.

Aziz and Shah [2] relaxed the hypothesis of Theorem A and proved the following result:

Theorem B: Let $f(z)=\sum_{j=0}^{\infty} a_j z^j (\neq 0)$ be analytic for $|z|\le 1$ and for some $k \ge 1$,

$$ka_0 \ge ta_1 \ge t^2 a_2 \ge ....,$$

then f(z) does not vanish in

$$\left| z - \frac{k-1}{2k-1} t \right| \le \frac{kt}{2k-1}.$$

Aziz and Zargar [4] generalised the above results by proving the following results:

Theorem C: Let $f(z)=\sum_{j=0}^{\infty} a_j z^j (\neq 0)$ be analytic for $|z|\le t$ and for some $k \ge 1$,

$$\max_{|z|=1} \left| (ka_0 - ta_1) + (a_1 - ta_2) z + (a_2 - ta_3) + ... \right| \le M.$$

Then f(z) does not vanish in the disk

$$\left| z - \frac{(k-1)|a_0|^2 t}{M^2 - (k-1)^2 |a_0|^2} \right| \le \frac{Mt|a_0|}{M^2 - (k-1)^2 |a_0|^2}.$$

Theorem D: Let $f(z)=\sum_{j=0}^{\infty} a_j z^j (\neq 0)$ be analytic for $|z|\le t$ and for some $k \ge 1$,

$$\max_{|z|=1} \left| (ka_0 - ta_1) + (a_1 - ta_2) z + (a_2 - ta_3) + ... \right| \le M.$$

Then f(z) does not vanish in

$$|z| \le \frac{t|a_0|}{(k-1)|a_0| + M}.$$

Theorem E: Let $f(z)=\sum_{j=0}^{\infty} a_j z^j (\neq 0)$ be analytic for $|z|\le R$ such that for some $k \ge 1$ and t>0,

$$\max_{|z|=R} |H(z)| \le M,$$

where

H(z) = $(ka_0 - ta_1) + (a_1 - ta_2) z + (a_2 - ta_3) + ...$.

Then f(z) does not vanish in

$$|z| \le \min\left\{ \frac{t|a_0|}{(k-1)|a_0| + M}, t \right\}.$$

In this paper we prove the following generalizations of the theorems C , D and E:

Theorem 1: Let $f(z)=\sum_{j=0}^{\infty} a_j z^j (\neq 0)$ be analytic for $|z|\le t$ and for some $\rho \ge 0$,

$$\max_{|z|=1} \left| (\rho + a_0 - ta_1) + (a_1 - ta_2) z + (a_2 - ta_3) + ... \right| \le M.$$

Then f(z) does not vanish in the disk

$$\left|z-\frac{\rho t|a_0|}{M^2-\rho^2}\right|\le\frac{Mt|a_0|}{M^2-\rho^2}.$$

Remark 1: Taking $\rho=(k-1)a_0$, Theorem 1 reduces to Theorem C.
Taking t=1, Theorem 1 immediately gives the following result:

Corrolary 1: Let $f(z)=\sum_{j=0}^{\infty}a_jz^j(\ne0)$ be analytic for $|z|\le1$ such that for some $\rho\ge0$,

$$\max_{|z|=1}|(\rho+a_0-a_1)+(a_1-a_2)z+(a_2-a_3)+.........|\le M.$$

Then f(z) does not vanish in

$$\left|z-\frac{\rho|a_0|}{M^2-\rho^2}\right|\le\frac{M|a_0|}{M^2-\rho^2}.$$

Theorem 2: Let $f(z)=\sum_{j=0}^{\infty}a_jz^j(\ne0)$ be analytic for $|z|\le t$ and for some $\rho\ge0$,

$$\max_{|z|=1}|(\rho+a_0-ta_1)+(a_1-ta_2)z+(a_2-ta_3)+.........|\le M.$$

Then f(z) does not vanish in

$$|z|\le\frac{t|a_0|}{\rho+M}.$$

Remark 2: Taking $\rho=(k-1)a_0$, Theorem 1 reduces to Theorem D.
Taking t=1, Theorem 1 immediately gives the following result:

Corrolary2: Let $f(z)=\sum_{j=0}^{\infty}a_jz^j(\ne0)$ be analytic for $|z|\le1$ and for some $\rho\ge0$,

$$\max_{|z|=1}|(\rho+a_0-ta_1)+(a_1-ta_2)z+(a_2-ta_3)+.........|\le M.$$

Then f(z) does not vanish in

$$|z|\le\frac{|a_0|}{\rho+M}.$$

Remark 3: If $f(z)=\sum_{j=0}^{\infty}a_jz^j(\ne0)$ be analytic for $|z|\le1$, $a_j>0$ and $a_{j-1}\ge ta_j$, $j=1,2,3,........$, then

$$\max_{|z|=t}|(\rho+a_0-ta_1)+(a_1-ta_2)z+(a_2-ta_3)z^2+......|$$

$$\le\rho+|a_0-ta_1|+|a_1-ta_2|t+|a_2-ta_3|t^2+......$$

$$=\rho+a_0-ta_1+ta_1-t^2a_2+t^2a_2-t^3a_3+......$$

$$=\rho+a_0$$

and we immediately get the following result from Theorem 1:

Corrolary3: Let $f(z)=\sum_{j=0}^{\infty}a_jz^j(\ne0)$ be analytic for $|z|\le t$ such that for some $\rho\ge0$,

$$\rho+a_0\ge ta_1\ge t^2a_2\ge.........$$

Then f(z) does not vanish in

$$|z|\le\frac{t|a_0|}{2\rho+a_0}.$$

Theorem 3: Let $f(z)=\sum_{j=0}^{\infty}a_jz^j(\ne0)$ be analytic for $|z|\le R$ such that for some $\rho\ge0$ and t>0,

$$\max_{|z|=R}|H(z)|\le M,$$

where

H(z) = $(\rho+a_0-ta_1)+(a_1-ta_2)z+(a_2-ta_3)+....$.

Then f(z) does not vanish in

$$|z|\le\min\{\frac{t|a_0|}{\rho+M},R\}.$$

Remark 4: Since

$$M=\max_{|z|=t}|H(z)|\ge|H(t)|$$

$$=|(\rho+a_0-ta_1)+(a_1-ta_2)t+(a_2-ta_3)t^2+......|$$

$$=|\rho+a_0|$$

$$\ge|a_0|-\rho,$$

we have

$$|a_0|\le\rho+M$$

so that

$$\frac{t|a_0|}{\rho+M}\leq t.$$

This shows that Theorem 2 is a special case of Theorem 3 for R=t .

Remark 5: Taking $\rho=(k-1)a_0$, Theorem 2 reduces to Theorem E.

Taking $\rho=0$ in Theorem 2, we get the following result:

Corrolary4: Let $f(z)=\sum_{j=0}^{\infty}a_j z^j (\neq 0)$ be analytic for $|z|\leq R$ and

$$\max_{|z|=R}\left|\sum_{j=1}^{n}(a_{j-1}-ta_j)z^{n-j}\right|\leq M.$$

Then f(z) does not vanish in

$$|z|\leq \min\{\frac{t|a_0|}{M},R\}.$$

Cor.4 was independently proved by Aziz and Shah [3] .

II. Proofs of Theorems

Proof of Theorem 1: It is obvious that $\lim_{j\to\infty} a_j z^j=0.$ Consider the function

$$F(z)=(z-1)f(z)$$

$$=(z-1)(a_0+a_1z+a_2z^2+......)$$

$$=-a_0+(a_0-a_1)z+(a_1-a_2)z^2+......$$

$$=-a_0-\rho z+(\rho+a_0-a_1)z+(a_1-a_2)z^2+......$$

$$=-a_0-\rho z+zH(z)$$

where

H(z) = $(\rho+a_0-a_1)+(a_1-a_2)z+(a_2-a_3)+....$.

Clearly

$$M=\max_{|z|=1}|H(z)|$$

$$\geq|H(1)|$$

$$=|(\rho+a_0-a_1)+(a_1-a_2)z+(a_2-a_3)+.....|$$

$$=|\rho+a_0| \qquad (1)$$

Since H(z) is analytic for $|z|\leq 1$ and $|H(z)|\leq M$ for $|z|=1$, by the Maximum Modulus Theorem , it follows that, $|H(z)|\leq M$ for $|z|\leq 1$. Therefore , for $|z|\leq 1$,

$$|F(z)|\geq|a_0+\rho z|-|z||H(z)|$$

$$\geq|a_0+\rho z|-M|z|$$

$$>0 \text{ if }$$

$$|a_0+\rho z|>M|z|,$$

which is true if

$$|a_0|+\rho|z|>M|z|$$

or

$$|z|<\frac{|a_0|}{M-\rho}.$$

It is easy to verify that the region E defined by

$$\{z;|z|<\frac{|a_0|}{M-\rho}\}$$

is precisely the disk

$$\{z;\left|z-\frac{\rho|a_0|}{M^2-\rho^2}\right|\leq\frac{t|a_0|}{M^2-\rho^2}\}.$$

Further , if $z\in E$, then $|z|<\frac{|a_0|}{M-\rho}\leq 1$ if $|a_0|\leq M-\rho$ i.e., $M\geq\rho+|a_0|$, which is true by (1). Using these observations we conclude that F(z) does not vanish in the disk

$$\left|z-\frac{\rho|a_0|}{M^2-\rho^2}\right|\leq\frac{t|a_0|}{M^2-\rho^2}.$$

Since the zeros of f(z) are also the zeros of F(z) , we conclude that f(z) does not vanish in the disk

$$\left|z-\frac{\rho|a_0|}{M^2-\rho^2}\right|\leq\frac{t|a_0|}{M^2-\rho^2}.$$

That completes the proof of Theorem 1.

Proof of Theorem 2: Consider the function

$$F(z)=(z-t)f(z)$$

$$=(z-t)(a_0+a_1z+a_2z^2+......)$$

$$=-a_0t+(a_0-ta_1)z+(a_1-ta_2)z^2+......$$

$$=-a_0t-\rho z+(\rho+a_0-ta_1)z+(a_1-ta_2)z^2+......$$

$$=-a_0t-\rho z+zH(z)$$

where

H(z) = $(\rho+a_0-ta_1)+(a_1-ta_2)z+(a_2-ta_3)+....$.

We first suppose that $M > \frac{t|a_0|}{R} - \rho$ i.e. $\frac{t|a_0|}{R} < \rho + M$ or

$$\frac{t|a_0|}{\rho + M} < R. \qquad (2)$$

Then since H(z) is analytic for $|z| \le R$ and $|H(z)| \le M$ for $|z| = R$, it follows , by Maximum Modulus Theorem, that $|H(z)| \le M$ for $|z| \le R$. Therefore, for $|z| \le R$,

$$\begin{aligned}|F(z)| &= |-a_0 t - \rho z + zH(z)| \\ &\ge |a_0 t + \rho z| - |z||H(z)| \\ &\ge t|a_0| - \rho|z| - M|z| \\ &\ge t|a_0| - (\rho + M)|z| \\ &> 0 \text{ if}\end{aligned}$$

$$|z| < \frac{t|a_0|}{\rho + M}$$

or if, by (2),

$$|z| < \frac{t|a_0|}{\rho + M} < R.$$

Thus $|F(z)| > 0$ for $|z| \le R$ i.e. $F(z) \ne 0$ for $|z| \le R$, if (2) is satisfied.

Now, suppose that $M < \frac{t|a_0|}{R} - \rho$ i.e. $\frac{t|a_0|}{R} > \rho + M$ or

$$\frac{t|a_0|}{\rho + M} > R. \qquad (3)$$

Then

$$\begin{aligned}|F(z)| &= |-a_0 t - \rho z + zH(z)| \\ &\ge t|a_0| - (\rho + M)R \\ &> 0 \quad \text{by using (3).}\end{aligned}$$

Thus $|F(z)| > 0$ for $|z| \le R$. This implies $F(z) \ne 0$ for $|z| \le R$. Hence, it follows that F(z) does not vanish for $|z| \le R$ in this case also.

Combining the above two arguments, we conclude that F(z) and hence f(z) does not vanish in the disk

$$|z| \le \min\{\frac{t|a_0|}{\rho + M}, R\}.$$

That proves the result.

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