Modal Decomposition Methods on Aerodynamic Flows

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Abstract- This paper discusses the significance of the modal decomposition methods that can be applied to identify dynamics in time-invariant systems specific to aerodynamic applications. Proper Orthogonal Decomposition can act on huge amount of data to detect patterns at low computational stress. Dynamic Mode Decomposition enables prediction of the future state with error developing over time while Koopman mode will result in null error in case the right observables are chosen. Research is ongoing for complex flows like flow past a cylinder where different modes of behavior appear depending on the Reynolds number. We also look into the Koopman Eigenfunctions germane to the attracting and asymptotic dynamics of the flow past a cylinder.

Index Terms-Proper Orthogonal Decomposition,Koopman Mode Decomposition, Burgers' Equation, Cylinder Wake, Least Squares

Nomenclature- Variables are defined locally for each section

I. Introduction

With the advent of data, there have been opportunities to apply formalism to detect patterns or simple relations. For instance, a phenomenon can be defined through a partial differential equation which may not be very useful right away whereas as a formula for the evolution of a primary variable may be interpreted quite easily. Having access to data is not enough to move on, since doing linear algebra can put strain on the way computations are being done.The idea behind Proper Orthogonal Decomposition came up in 1970 (Lumley) and then in 1987 (Sirovich). In 2005, Rowley introduced Balanced POD as an improvement to the actual POD: this extended version accounted for some of the important features and dealt with the error concerning any truncation for linear systems. Schmid (2010) used Dynamic Mode Decomposition to explain that the resultant eigenvalues and eigenvectors are related to the progress of the flow over time. Rowley along with other researchers discovered the connection of this DMD to the Koopman operator. The flow past a cylinder is a canonical flow in fluid mechanics that sees significant change in dynamics over time based on the Reynolds number governed by velocity, air viscosity and the diameter of the cylinder. There would be symmetrical vortical structures for $Re < Re_{cr}$. We shall observe the dynamics slightly post critical Reynolds number ($Re_{cr} = 46.6$) when the oscillation in the wake just begins. The reason being the modes in that regime relate to the Stuart-Landau Model that will lead to the observables we are after. Note that the end goal is to represent non-linear dynamics in a linear form.

II. Preliminaries

Proper Orthogonal Decomposition

POD is a model-reduction technique in the statistical analysis of vector data. Often times, we have data for a given state variable (e.g. velocity) at each grid point (q) at different time instances (p). Assume that they are all stored in a $p\ X\ q$ matrix named A according to [3]. Singular Value Decomposition of A results in U, Σ, V. U and V are orthonormal matrices of size $p\ X\ p$ and $q\ X\ q$ respectively.

$$A = U\Sigma V^T \tag{1}$$

Σ is the diagonal matrix containing the singular values in the diagonal. The rank of Σ, r, is the number of non-zero singular values. An optimal rank $k < r$ approximation can be found by

$$A_k = U\Sigma_k V^T \tag{2}$$

Dynamic Mode Decomposition

This tool extracts modes off data and can predict future states. Let's say, there is a velocity matrix u of dimension $N_x\ X\ N_t$ where each row corresponds to each coordinate in the domain and each column is for a particular time instant. Then, applying the following algorithm gives an approximation to retrieve and approximate the solution in future times.

Algorithm 1 Dynamic Mode Decomposition

```
[U, S, W] = svd(u(:, 1 : N_t − 1),′, econ)
R = U′ ∗ u(:, 2 : N_t) ∗ W ∗ inv(S)
[evec, eval] = eig(R)
modes = U ∗ evec
b = pinv(modes) ∗ u(:, 1)
ω = log(diag(eval))/dt
u_dmd = Real(modes ∗ b ∗ exp(ω ∗ t)
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Koopman Mode Decomposition

This approach follows the same pseudocode as Dynamic mode decomposition, but works in the observable space. The data would be extended to contain a few non-linear functions

$$u = [u; \sqrt{u}; |u^{1.5}|] \tag{3}$$

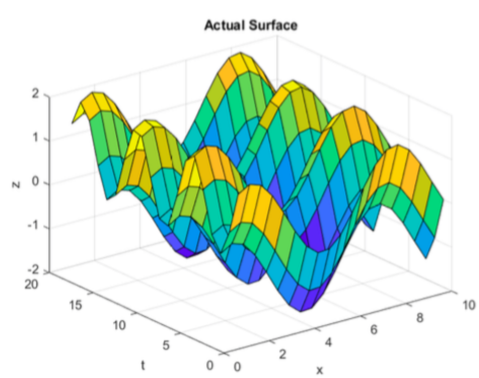
It is not an easy task to decide a priori which features to pick to run the algorithm. This method proves to be much more accurate than Dynamic Mode Decomposition. The inner details can be better understood by considering the following dynamical system in Eq.(4) that can be written as in Eq.(5) if $g = e^{-\frac{1}{x}}$ derived using Laurent and Taylor Series.

$$\frac{dx}{dt} = \lambda x^2 \tag{4}$$

$$\frac{dg}{dt} = \lambda g \tag{5}$$

III. Numerical Experiment

A. Wavelet $(z(x,t) = sin(x) + cos(t))$ As evident in Figure 1, Rank 2 approximation is better than the rank 1 approximation. Figure 2 illustrates that the first two modes contain most of the energy.



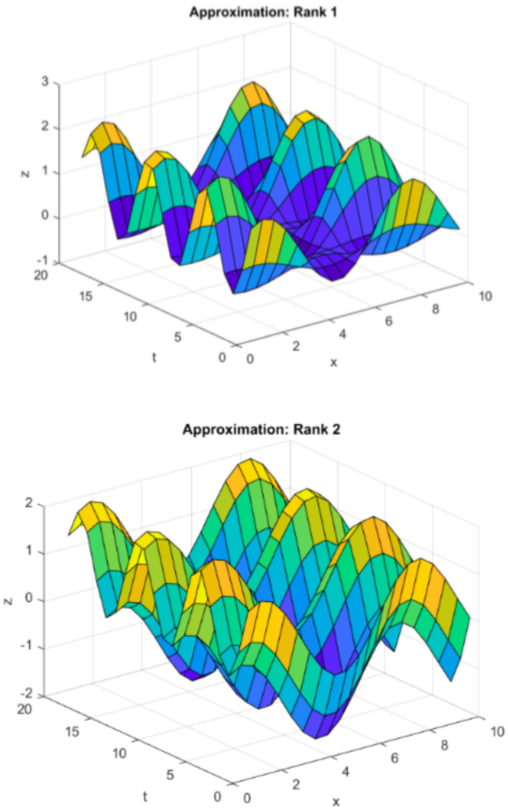


Figure 1. Wavelet and Approximations

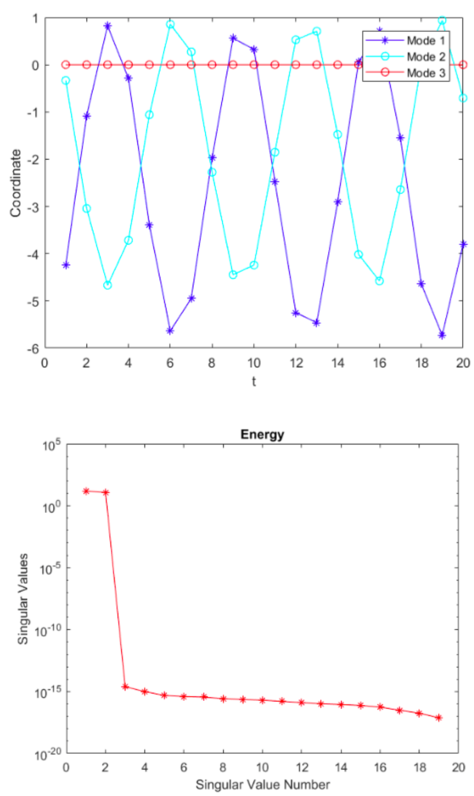


Figure 2. Eigenmodes and Energy content

B. Burgers' Equation (A hyperbolic non-linear PDE in fluids)

Data are collected through finite difference on the PDE for upto t=1 and we try to estimate the solution at t=3 ($\epsilon = 0.1$). Dynamic Mode Decomposition will see error growing over time, whereas koopman mode is perfect since it uses $\hat{v}$ where v is given by Eq(7) by Cole-Hopf seminal work. Figure 3 displays the results. The modeled solution agree well with the numerical solution.

$$u_t = uu_x - \epsilon u_{xx} = 0, x \ \epsilon \ [0, 2\pi] \tag{6}$$

$$v(x,t) = e^{\frac{-\int_{-\infty}^{x} u(\xi,t)d\xi}{2\epsilon}} \tag{7}$$

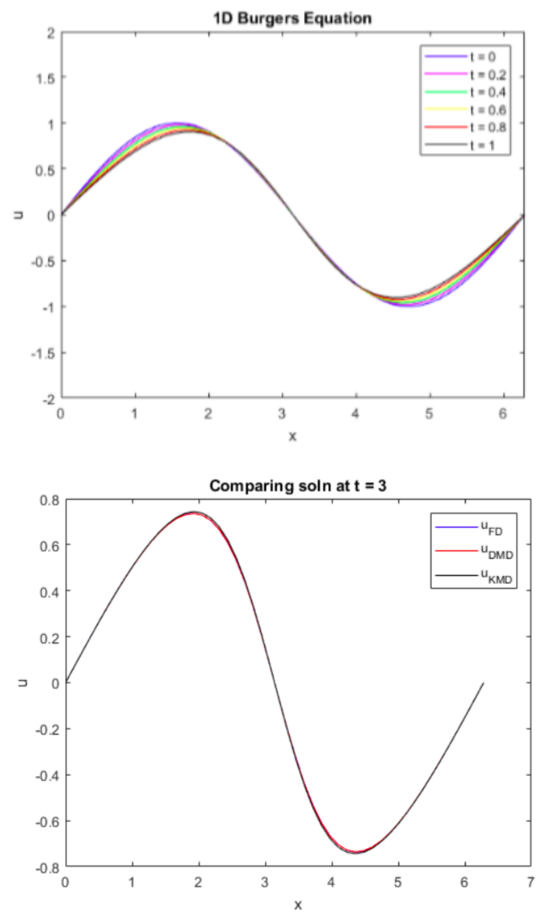


Figure 3. Burgers' Equation solution and approximate future states

C. Post-critical Reynolds number flow past a cylinder The flow past a cylinder is a canonical flow in fluid mechanics that exhibits influence by Reynolds number, $Re = \frac{VD}{\nu}$, (a function of inflow velocity (V), diameter of the cylinder and the kinematic viscosity (ν)). The conservation of mass and momentum relating to the 2D velocity field are

$$Continuity : \nabla.\mathbf{u} = 0 \tag{8}$$

$$Momentum : \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u} = \frac{\nabla p}{\rho} + \nu\nabla^2\mathbf{u} \tag{9}$$

When problems are stated in terms of the PDE, not all of them have smooth consistent solution over time. For example, Burger's equation will admit shock at some point of time. Such perturbation problems can be dealt with what is called multiple scale expansion method. It introduces an additional independent variable (let's say if t is the independent variable, then τ is the new independent variable: $\tau = \epsilon t$). Another useful part of the machinery in this analysis is the Stuart-Landau model to describe the Hopf-bifurcation

$$\frac{dA}{dt} = (\mu + i\gamma)A - (1 + i\beta)|A|^2 A + H.O.T \tag{10}$$

As per [1], parameters μ, γ, β are real and A is a complex values function of time t (e.g. local quantity in a flow). We generally have odd terms on RHS.

$$A(t) = \rho(t)e^{i\phi(t)} \tag{11}$$

ρ = amplitude

ϕ = phase

Injecting (11) into (10) gives:

$$\rho^{'} = \mu\rho - \rho^3 \tag{12}$$

$$\phi^{'} = \gamma - \beta\rho^2 \tag{13}$$

In order to quantify S-L model parameters, simulation can be run to record the local ρ and ϕ over time and derivatives be taken by central differencing. The command *lsqcurvefit* on MATlab performs the Non-linear Least Squares. FLUENT solves the flow at around Re of 50 to observe the non-dimensionalized drag over time to confirm that the drag coefficient moved into the limit cycle.The system $\frac{du}{dt} = (\mu + i\gamma)u$ means that $(\mu + i\gamma)$ is the eigenvalue of the linearized Navier-Stokes operator where μ= the amplification rate and γ= angular frequency. These two are global in the sense that they hold the same values at all the spatial locations. We will find out, however, that β is a function of space.

In [2], the authors used Nekton code to perform the simulation to get the transverse velocity profile at Re=48 around the infinite cylinder. The BCs are a uniform inflow velocity at the left boundary, a periodicity condition at the lateral boundaries and a free outflow condition at the right boundary. The points selected are from the upstream and downstream locations and most of them lying in the wake centerline to examine the instability threshold. The time evolution of the transverse velocity in the upstream and downstream of the cylinder for $y = 0$ hovers around a value and sees exponential growth whereas the signal is asymmetric for non-zero y coordinates. There are three real constants: amplification rate, angular frequency and the change in the angular frequency at the saturation that are significant in characterizing the unstable mode.

With the knowledge of the S-L model parameters, we make an attempt to the derive the Koopman eigenfunction and the Koopman eigenvalues. This helps estimating the flow as

$$u(t) = \sum_{j=0}^{\infty} \phi_j \ \mathbf{v}_j e^{[(\sigma_j + i\omega_j)t]}$$

where $\sigma_j + i\omega_j = \lambda_j$

ϕ = Koopman eigenfunction or amplitude

v =mode

The koopman eigenfunctions are found to be given by

$$\phi_{j,m} = \sum_{m=-\infty}^{\infty} \hat{\phi}_{j,m}(\rho)e^{im\theta}$$

$\rho =$ amplitude

$\theta =$ phase of the signal.

As manifest as it is, discrete Fourier transform is the key to get information about the eigenfunction. We observe that the observable would be a linear combination of a steady term and a series of exponentially decaying non-linear functions of the amplitude.

$$g(A) = \left|A\right|^2 = \mu - \mu(\frac{\mu}{\rho_0^2} - 1)e^{-2\mu\tau} + ... \tag{14}$$

$$\frac{dg}{d\tau} = \frac{d\rho}{d\tau}\frac{dg}{d\rho} + \frac{d\theta}{d\tau}\frac{dg}{d\theta} = Lg$$

$$\mathbf{g}(\mathbf{u}(\tau)) = e^{(L\tau)}\mathbf{g}(\mathbf{u}_0) \tag{15}$$

The eigenfunctions are non-linear functions of initial amplitude and the model parameters and the eigenvalues are sensitive only to the model parameters.

$$\phi_{j,m} = (\frac{\mu}{\rho_0^2} - 1)^j e^{[im(\theta_0 + \beta ln(\frac{\mu^{0.5}}{\rho_0}))]} \tag{16}$$

$$\hat{\lambda}_{j,m} = -2j\mu + im(\gamma - \mu\beta) \tag{17}$$

where 0 denotes initial solution

$m = 0, \pm 1, \pm 2......$

IV. Conclusion

This work clarifies the concepts and application of the mode decomposition methods.Some can reconstruct and others would be employed as a solver upto certain point in time. We realize that analytical estimation of observable from PDE is very difficult and requires a posteriori knowledge! Derivations are also shown to search for Koopman observables for flow past a cylinder in the attracting and the asymptotic regime.

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VI. Reference

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