A discussion and comparison between Circular and Schrödinger Waves

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Abstract- This study presents the comparison between circular and Schrödinger waves. The Hankel functions $H_0^{(1)}(mr)$ and $H_0^{(2)}(mr)$ which are the solution of circular waves (sink) when combined with e^{-ikt} and diverging wave (source) when combined with e^{ikt} respectively. Moreover the functions $J_0(mr)$ and $Y_0(mr)$ of $x = mr$ suggest that each have a damped oscillatory behavior as $x \to \infty$ and that the positive and zeros of J_0 and Y_0 separate each other. On the other hand the Schrödinger wave equation represent that the positive $(+)$ sign is a wave traveling in the $+x$ direction and the negative $(-)$ sign for traveling in the $-x$ direction. In this study the authors have tried to find out the comparison between the circular wave and Schrödinger wave on the basis of monochromatic plane wave.

Index Terms- Bessel function, Circular wave, Hankel function and Schrödinger wave

I. Introduction

The Laplace equation is often used for nonlinear wave propagation both in shallow and deep water. This equation has been treated in circular wave motion. Several investigators have previously worked on the mathematical equation, which describes the motion of water waves (e.g. Dean and Dalrymple; Hoque; Parvin) [1, 2, 3]. Keller [4] developed a set of equations describing the evolution of two interacting wave components. He demonstrated that, in the non-dispersive limit, this same set of evolution equations could be derived from the exact Euler equations, the nonlinear shallow water equations, and the Boussinesq equations (J. M. Kaihatu and J. T. Kirby.) [5].

In physics, a field is a physical quantity associated to each point of space-time. The concept of a quantum field is very wide, embracing all physical quantities which depend on space and time. A free field is a field whose equations of motion are given by linear partial differential equations. The actual formalism of quantum field theory is partly based on a suitable generalization of the one already used for systems of point particles (Mukul Agrawal)[6]. Laws of thermodynamics and classical laws of electricity and magnetism provide the basis for explanation of all phenomena in classical physics (W.Thirring) [7]. In 1926, Austrian physicist Erwin Schrödinger reasoned that if electrons behave as waves, then it should be possible to describe them using a wave equation, like the equation that describes the vibrations of strings, or Maxwell's equation for electromagnetic waves. The time-dependent Schrödinger equation or wave equation for a single non-relativistic charged particle moving in an electric field describes all the features of the electron that we can measure, and can be extended to include any other object under almost any other force. Many physicists preferred Schrödinger's approach because it was easier to visualize and used more familiar mathematics. Erwin Schrödinger (Austrian) constructs a wave equation for de Broglie's matter waves. Schrödinger's works with a wave function Ψ purely for mathematical convenience. He expects that, in the end, he will take the real part of Ψ to get the physically "real" matter wave. The concept of a monochromatic plane wave is an idealization and does not represent a real physical situation. Schrodinger wave equation represent that the positive sign is a wave traveling in the positive x direction and the negative sign for traveling in the negative x direction.

II. Research Elaborations

Circular wave's equation

The fluid motion can be described by a velocity potential, which is governed by the Laplace equation in terms of cylindrical coordinates (r, θ, z):

$$\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} + \phi_{zz} = 0 \qquad (1)$$

In the case of a motion with an axis of symmetry so that $\phi_{\theta\theta} = 0$, equation (1) becomes

$$\phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} = 0 \qquad (2)$$

Solution of the Circular wave equation

Let $\phi(r, z, t) = U(r).P(z).f(t)$ be the trial solution of equation (2), the cylindrical wave becomes

$$U_{rr} + \frac{1}{r}U_r + m^2U = 0 \qquad (3)$$

$$U = AJ_0(mr) + BY_0(mr); m \neq 0 \qquad (4)$$

Again, the Bessel's differential equation in Cartesian coordinates of n order is

$$x^2y_{xx} + xy_x + (x^2 - n^2)y = 0 \qquad (5)$$

If we consider x is very large and $y = \frac{U}{\sqrt{x}}$, then the above equation becomes

$$U_{xx} + U = 0 \qquad (6)$$

This is the second order differential equation.

Substituting the value of U and putting $x = mr$, the general solution of equation (6) can be written as

$$y = A_1\frac{\cos mr}{\sqrt{mr}} + B_1\frac{\sin mr}{\sqrt{mr}} \qquad (7)$$

Again the general solution of equation (5) can be found as

$$y = C_1J_n(x) + C_2J_{-n}(x) \qquad (8)$$

For non-integral values of n, we have $J_n(x)$ from above equation as

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)}\left[1 - \frac{x^2}{2.2(n+1)} + \frac{x^4}{2.4.2^2(n+1)(n+2)} - \cdots\right]$$

Using these values in equation (8) with the help of $n = \pm(1/2)$ and $x = mr$,

Finally we have,

$$y = C_1\sqrt{\frac{2}{\pi mr}}\sin(mr) + C_2\sqrt{\frac{2}{\pi mr}}\cos(mr) \quad (9)$$

Now comparing equations (7)and(9), we obtain

$A_1 \approx \sqrt{2/\pi}$ and $B_1 \approx \sqrt{2/\pi}$.

Using these values in equation(7), it can be written as

$$y = \sqrt{\frac{2}{\pi mr}}\cos mr + \sqrt{\frac{2}{\pi mr}}\sin mr \quad (10)$$

For large value of mr and putting $n = 0$, the above relations reduce to

$$J_0(mr) = A\cos B \quad (11a)$$

and $$Y_0(mr) = A\sin B \quad (11b)$$

Where $A = \sqrt{2/\pi mr}$ and $B = (mr - \pi/4)$.

Solutions of peculiar interest are given by Hankel functions $H_0^{(1)}(mr)$ and $H_0^{(2)}(mr)$ which are defined as

$$H_0^{(1)}(mr) = J_0(mr) + iY_0(mr) \quad (12a)$$

and $$H_0^{(2)}(mr) = J_0(mr) - iY_0(mr) \quad (12b)$$

These implies with the help of equations(11)and(12), respectively

$$H_0^{(1)}(mr) \approx A\cos B + i\sin B \approx Ae^{iB} \quad (13a)$$

and $$H_0^{(2)}(mr) \approx A\cos B - i\sin B \approx Ae^{-iB} \quad (13b)$$

Schrödinger Wave Equation

The Schrödinger equation is the fundamental equation of physics for describing quantum mechanical behavior. It is also often called the Schrödinger wave equation, and is a partial differential equation that describes how the wave function of a physical system evolves over time. Viewing quantum mechanical systems as solutions to the Schrödinger equation is sometimes known as Schrödinger picture.

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi$$

Solution of the Schrödinger wave equation

We know the time dependent Schrodinger wave equation is

$$i\hbar\frac{\partial}{\partial t}\psi(\bar{r},t) = H\psi(\bar{r},t) \quad (14)$$

Where H is a differential operator called the Hamiltonian. For a single particle of mass m moving in a scalar potential energy field $U(\bar{r},t)$, the Hamiltonian is

$$H = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + U(\bar{r},t) = -\frac{\hbar^2}{2m}\nabla^2 + U(\bar{r},t) \quad (15)$$

and the normalization condition (probability of finding the particle is one) is

$$\int_v \Psi(\bar{r},t)^*\Psi(\bar{r},t)dV = 1 \quad (16)$$

The probability density ρ associated with the single particle is

$$\rho(\bar{r},t) = \Psi(\bar{r},t)^*\Psi(\bar{r},t) \quad (17)$$

and using equations (16) and (17), the time rate of change of the probability density gives an equation of continuity

$$\frac{\partial\rho}{\partial t} = \frac{\partial\Psi^*}{\partial t}\Psi + \Psi^*\frac{\partial\Psi}{\partial t} = \frac{i\hbar}{2m}\nabla^2(\Psi^*\nabla\Psi) - \Psi\nabla\Psi^*)\frac{\partial\rho}{\partial t} + \nabla^2\bar{J} = 0 \quad (18)$$

Where $\bar{J}$ is the probability current density.

$$\bar{J} = \frac{i\hbar}{2m}\nabla^2(\Psi^*\nabla\Psi) - \Psi\nabla\Psi^*) \quad (19)$$

If $\Psi(\bar{r},t)$ defines a pureenergy or stationarystate where the total energy of the particle is E then

$$\Psi(\bar{r},t) = \psi(\bar{r})e^{-i\frac{Et}{\hbar}} \quad (20)$$

and the time dependent Schrödinger equation (14) reduces to the time independent Schrödinger equation

$$H\psi(\bar{r}) = E\psi(\bar{r}) \quad (21)$$

Again, for a single particle moving along the x-axis as a free particle with total energy E and zero potential energy, U(x) = 0, the time dependent Schrödinger equation is

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) \quad (22)$$

The time operator $i\hbar\frac{\partial}{\partial t}$ defines the total energy of the particle E (energy Eigen state) and the spatial operator $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$ defines the momentum of the particle $p = \sqrt{2mE}$ (momentum Eigen state).

The solution of equation (22) can be expressed as a monochromatic plane wave by

$$\Psi(x,t) = Ae^{i(\pm kx - \omega t)} = A[\cos(kx - \omega t) \pm i\sin(kx - \omega t)] \quad (23)$$

III. Results

We have found the relationships between J_0, Y_0 and the Hankel functions $H_0^{(1)}(mr)$and $H_0^{(2)}(mr)$as well as their physical meanings and their analogy with sinusoidal functions. The appropriate solution and its physical significance is obtained by combining the solution with e^{-ikt} or e^{ikt} and examining its asymptotic form at large distance from the origin [Fig.1(c)].The Hankel functions $H_0^{(1)}(mr)$ and $H_0^{(2)}(mr)$ solutions of circular waves show the converging wave (sink) when combined with $e^{-ikt}$and diverging wave (source) when combined withe^{ikt}, respectively. On the other hand the Schrödinger wave equation [Fig.2] represent the positive $(+)$ sign is a wave traveling in the $+x$ direction and the negative $(-)$ sign for traveling in the $-x$ direction.

FIGURE CAPTIONS and Program with MATLAB

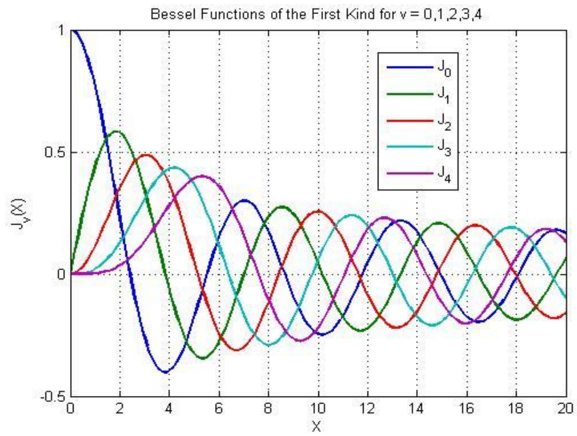
Fig. 1(a): The graph of functions Ji (i = 0, 1, 2, 3, 4)

Fig. 1(b): The graph of functions Yi (i = 0, 1, 2, 3, 4)

Fig. 1(c): The combined graphs of functions Jo and Yo.

Fig. 2: The graphs of Schrödinger wave function.

Fig. 3: Comparison the graphs between Circular wave and Schrödinger wave function.



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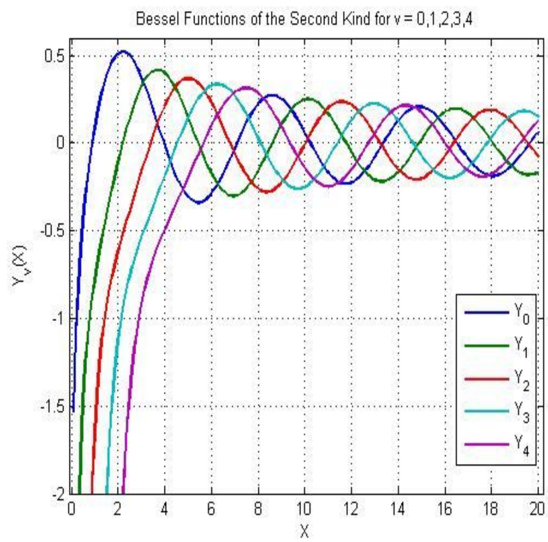
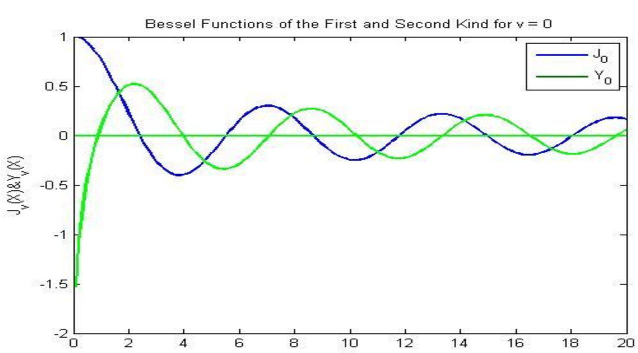


Fig. 1(b): The graph of functions Yi (i = 0, 1, 2, 3, 4)



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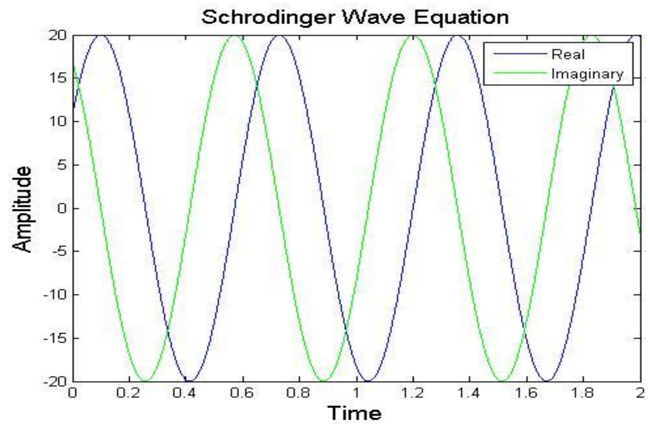

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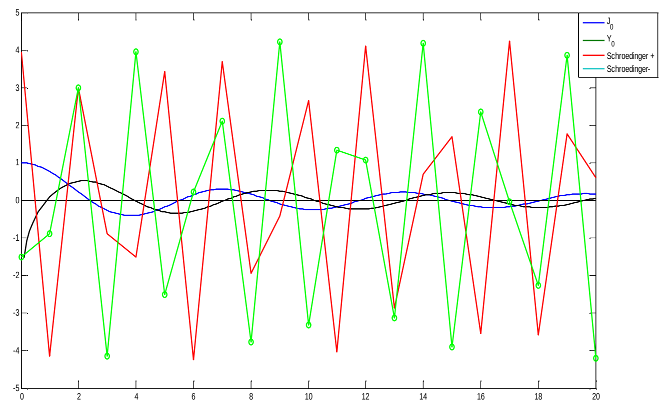

Fig. 3: Comparison the graphs between Circular wave and Schrödinger wave function.

IV. Conclusions

The standard solutions of Bessel equation of order zero are denoted by $J_0(mr)$ and$Y_0(mr)$, the first being regular at the origin, the second singular at the origin. The functions J_0 and Y_0 have been studied extensively. Many of the interesting properties of these functions are indicated by their graph, which are shown in [Fig.1(a, b, c)]. For one thing, they suggest that J_0 and Y_0 each have a damped oscillatory behavior as $x \to \infty$ and that the positive and negative zeros of J_0 and Y_0 separate each other. Any linear combination of these standards solution is itself obviously a solution of the equation. The combination of the Schrödinger wave and circular wave [Fig.3] of a monochromatic plane wave is an idealization and does not represent a real physical situation.

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