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OBSERVATIONS ON HALF-COMPANION PELL-NUMBERS

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Abstract- Pell number together with Half companion Pell number have been analysed. Some identities among these numbers are presented.

Index Terms- Pell number, Half companion Pell number.

I. Introduction

Number is the essence of Mathematical calculation. Varieties of numbers have variety of range and richness. Manyintegers exhibit fascinating properties, they form sequences, they form patterns and so on [1-21]. In this communication we consider twin sequences Pell numbers and Half companionPall numbers and some identities relating themselves.

II. PROPERTIES

1. $H_{n+2} = 3H_n + 4P_n$

Proof:

$$H_{n+2} = \frac{(1+\sqrt{2})^{n+2} + (1-\sqrt{2})^{n+2}}{2}$$

$$= \frac{(1+\sqrt{2})^n (3+2\sqrt{2}) + (1-\sqrt{2})^n (3-2\sqrt{2})}{2}$$

$$= 3\frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2} + 2\sqrt{2}\,\frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2}$$

$$= 3H_n + 4P_n$$

$$H_{n+2} = 3H_n + 4P_n$$

2. $P_{n+2} = 3P_n + 2H_n$

Proof:

$$P_{n+2} = \frac{(1+\sqrt{2})^{n+2} - (1-\sqrt{2})^{n+2}}{2\sqrt{2}}$$

$$= \frac{(1+\sqrt{2})^n (3+2\sqrt{2}) - (1-\sqrt{2})^n (3-2\sqrt{2})}{2\sqrt{2}}$$

$$= 3\frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}} + 2\sqrt{2}\,\frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2\sqrt{2}}$$

$$= 3P_n + 2H_n$$

$$P_{n+2} = 3P_n + 2H_n$$

3. $P_n^3 = \frac{1}{2^3}\left[P_{3n} - 3(-1)^n P_n \right]$

Proof:

$$P_n^3 = \left[\frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}} \right]^3$$

$$= \frac{1}{8}\left[\frac{(1+\sqrt{2})^{3n} - (1-\sqrt{2})^{3n}}{2\sqrt{2}} + 3(-1)^n \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}} \right]$$

$$= \frac{1}{8}\left[P_{3n} - 3(-1)^n P_n \right]$$

$$P_n^3 = \frac{1}{2^3}\left[P_{3n} - 3(-1)^n P_n \right]$$

4. $P_n^5 = \frac{1}{2^6}\left[P_{5n} - 5(-1)^n P_{3n} + 10P_n \right]$

Proof:

$$P_n^5 = \left[\frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}} \right]^5$$

$$= \frac{1}{128\sqrt{2}}\left[(1+\sqrt{2})^n - (1-\sqrt{2})^n \right]^5$$

$$= \frac{1}{64}\left[\frac{(1+\sqrt{2})^{5n} - (1-\sqrt{2})^{5n}}{2\sqrt{2}} - 5(-1)^n \frac{(1+\sqrt{2})^{3n} - (1-\sqrt{2})^{3n}}{2\sqrt{2}} + 10\frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}} \right]$$

$$= \frac{P_{5n} - 5(-1)^n P_{3n} + 10P_n}{64}$$

$$P_n^5 = \frac{1}{2^6}\left[P_{5n} - 5(-1)^n P_{3n} + 10P_n \right]$$

5. $P_n^7 = \frac{1}{2^9}\left[P_{7n} - 7(-1)^n P_{5n} + 21P_{3n} - 35(-1)^n P_n \right]$

Proof:

$$P_n^7 = \left[\frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}} \right]^7$$

$$= \frac{1}{2^9}\left[\begin{array}{l} \frac{(1+\sqrt{2})^{7n} - (1-\sqrt{2})^{7n}}{2\sqrt{2}} - 7(-1)^n \frac{(1+\sqrt{2})^{5n} - (1-\sqrt{2})^{5n}}{2\sqrt{2}} + 21\frac{(1+\sqrt{2})^{3n} - (1-\sqrt{2})^{3n}}{2\sqrt{2}} \\ -35(-1)^n \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}} \end{array} \right]$$

$$= \frac{1}{2^9}\left[P_{7n} - 7(-1)^n P_{5n} + 21P_{3n} - 35(-1)^n P_n \right]$$

$$P_n^7 = \frac{1}{2^9}\left[P_{7n} - 7(-1)^n P_{5n} + 21P_{3n} - 35(-1)^n P_n \right]$$

Remark:

From the relations 3, 4, and 5 a general representation for P_n^{2r+1} is obtained as follows:

$$P_n^{2r+1} = \frac{1}{2^{3r}}\left[P_{(2r+1)n} - (2r+1)C_1(-1)^n P_{(2r-1)n} + (2r+1)C_2 P_{(2r-3)n} - (2r+1)C_3(-1)^n P_{(2r-5)n} + \ldots\right]$$

Also the general representation for P_n^{2r} is

$$P_n^{2r} = \frac{1}{2^{3r-1}}\left[H_{2rn} - 2rC_1(-1)^n H_{(2r-2)n} + 2rC_2 H_{(2r-4)n} - 2rC_3(-1)^n H_{(2r-6)n} + \ldots + \frac{2rC_r}{2}(-1)^{k(n+3)-2}\right]$$

where $P_{-sn} = 0, \mathrm{s} = 0,1,2,3\ldots$

6. $H_n^3 = \dfrac{H_{3n} + 3(-1)^n H_n}{2^2}$

Proof:

$$H_n^3 = \left[\frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2}\right]^3$$

$$= \frac{1}{8}\left[(1+\sqrt{2})^n + (1-\sqrt{2})^n\right]^3$$

$$= \frac{1}{4}\left[\frac{(1+\sqrt{2})^{3n} + (1-\sqrt{2})^{3n}}{2} + 3(-1)^n \frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2}\right]$$

$$= \frac{H_{3n} + 3(-1)^n H_n}{4}$$

$$H_n^3 = \frac{H_{3n} + 3(-1)^n H_n}{2^2}$$

7. $H_n^5 = \dfrac{H_{5n} + 5(-1)^n H_{3n} + 10 H_n}{2^4}$

Proof:

$$H_n^5 = \left[\frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2}\right]^5$$

$$= \frac{1}{32}\left[(1+\sqrt{2})^n + (1-\sqrt{2})^n\right]^5$$

$$= \frac{1}{16}\left[\frac{(1+\sqrt{2})^{5n} + (1-\sqrt{2})^{5n}}{2} + 5(-1)^n \frac{(1+\sqrt{2})^{3n} + (1-\sqrt{2})^{3n}}{2} + 10\frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2}\right]$$

$$= \frac{H_{5n} + 5(-1)^n H_{3n} + 10 H_n}{16}$$

$$H_n^5 = \frac{H_{5n} + 5(-1)^n H_{3n} + 10 H_n}{2^4}$$

Remark:

From the relations 6 and 7a general representation for H_n^{2r+1} is obtained as follows:

$$H_n^{2r+1} = \frac{1}{2^{2r}}\left[H_{(2r+1)n} + (2r+1)C_1(-1)^n H_{(2r-1)n} + (2r+1)C_2 H_{(2r-3)n} + (2r+1)C_3(-1)^n H_{(2r-5)n} + \ldots\right]$$

Also the general representation for H_n^{2r}

$$H_n^{2r} = \frac{1}{2^{2r-1}}\left[H_{2rn} + 2rC_1(-1)^n H_{(2r-2)n} + 2rC_2 H_{(2r-4)n} + 2rC_3(-1)^n H_{(2r-6)n} + ... + \frac{2rC_r}{2}(-1)^{k(n+2)-2}\right]$$

where $H_{-sn} = 0, s = 0,1,2,3...$

8. $H_{2n} = 2H_n^2 - (-1)^n$

Proof:

$$H_{2n} = \frac{(1+\sqrt{2})^{2n} + (1-\sqrt{2})^{2n}}{2}$$

$$= \frac{\left[(1+\sqrt{2})^n + (1-\sqrt{2})^n\right]^2 - 2(1+\sqrt{2})^n(1-\sqrt{2})^n}{2}$$

$$= 2\left[\frac{\left[(1+\sqrt{2})^n + (1-\sqrt{2})^n\right]}{2}\right]^2 - (-1)^n$$

$$= 2H_n^2 - (-1)^n$$

$$H_{2n} = 2H_n^2 - (-1)^n$$

9. $H_{2n} = 4P_n^2 + (-1)^n$

Proof:

$$H_{2n} = \frac{(1+\sqrt{2})^{2n} + (1-\sqrt{2})^{2n}}{2}$$

$$= \frac{\left[(1+\sqrt{2})^n - (1-\sqrt{2})^n\right]^2 + 2(1+\sqrt{2})^n(1-\sqrt{2})^n}{2}$$

$$= 4P_n^2 + (-1)^n$$

$$H_{2n} = 4P_n^2 + (-1)^n$$

10. $P_{2n} - 2H_nP_n = 0$

Proof:

$$P_{2n} = \frac{(1+\sqrt{2})^{2n} - (1-\sqrt{2})^{2n}}{2\sqrt{2}}$$

$$= \left[(1+\sqrt{2})^n + (1-\sqrt{2})^n\right]\left[\frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}\right]$$

$$= 2H_nP_n$$

$$P_{2n} - 2H_nP_n = 0$$

11. $P_{2n} = 2P_n(H_{n+1} - 2P_n)$

Proof:

$$P_nH_{n+1} = \left[\frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}\right]\left[\frac{(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}}{2}\right]$$

$$= \frac{1}{4\sqrt{2}}\left[(1+\sqrt{2})^{2n} - (1-\sqrt{2})^{2n} + \sqrt{2}((1+\sqrt{2})^n - (1-\sqrt{2})^n)^2\right]$$

$$= \frac{1}{2}\left[\frac{(1+\sqrt{2})^{2n} - (1-\sqrt{2})^{2n}}{2\sqrt{2}} + \frac{\sqrt{2}((1+\sqrt{2})^n - (1-\sqrt{2})^n)^2}{2\sqrt{2}}\right]$$

$$=\frac{1}{2}\left[P_{2n}+4P_n^2\right]$$

$$P_{2n}=2P_n(H_{n+1}-2P_n)$$

12. $H_n^2+H_{n+1}^2=2(H_{2n}+P_{2n})$

Proof:

$$H_n^2+H_{n+1}^2=\left[\frac{(1+\sqrt{2})^n+(1-\sqrt{2})^n}{2}\right]^2+\left[\frac{(1+\sqrt{2})^{n+1}+(1-\sqrt{2})^{n+1}}{2}\right]^2$$

$$=\frac{1}{4}\left[(1+\sqrt{2})^{2n}+(1-\sqrt{2})^{2n}+2(-1)^n+3(1+\sqrt{2})^{2n}+2\sqrt{2}(1+\sqrt{2})^{2n}+3(1-\sqrt{2})^{2n}-2\sqrt{2}(1-\sqrt{2})^{2n}-2(-1)^n\right]$$

$$=\frac{1}{4}\left[4((1+\sqrt{2})^{2n}+(1-\sqrt{2})^{2n})+2\sqrt{2}((1+\sqrt{2})^{2n}-(1-\sqrt{2})^{2n})\right]$$

$$=\frac{1}{4}\left[8\frac{(1+\sqrt{2})^{2n}+(1-\sqrt{2})^{2n}}{2}+8\frac{(1+\sqrt{2})^{2n}-(1-\sqrt{2})^{2n}}{2\sqrt{2}}\right]$$

$$=2(H_{2n}+P_{2n})$$

$$H_n^2+H_{n+1}^2=2(H_{2n}+P_{2n})$$

13. $H_{n+1}=H_n+2P_n$

Proof:

$$H_{n+1}=\frac{(1+\sqrt{2})^{n+1}+(1-\sqrt{2})^{n+1}}{2}$$

$$=\frac{1}{2}\left[(1+\sqrt{2})^n+\sqrt{2}(1+\sqrt{2})^n+(1-\sqrt{2})^n-\sqrt{2}(1-\sqrt{2})^n\right]$$

$$=\frac{1}{2}\left[(1+\sqrt{2})^n+(1-\sqrt{2})^n+\sqrt{2}((1+\sqrt{2})^n-(1-\sqrt{2})^n)\right]$$

$$=H_n+2P_n$$

$$H_{n+1}=H_n+2P_n$$

14. $P_{2n}=H_n(H_{n+1}-H_n)$

Proof:

$$H_{n+1}H_n=\left[\frac{(1+\sqrt{2})^{n+1}+(1-\sqrt{2})^{n+1}}{2}\right]\left[\frac{(1+\sqrt{2})^n+(1-\sqrt{2})^n}{2}\right]$$

$$=\left[\frac{(1+\sqrt{2})^n+(1-\sqrt{2})^n}{2}\right]^2+\sqrt{2}\left[\frac{(1+\sqrt{2})^n+(1-\sqrt{2})^n}{2}\right]\left[\frac{(1+\sqrt{2})^n-(1-\sqrt{2})^n}{2}\right]$$

$$=H_n^2+P_{2n}$$

$$P_{2n}=H_n(H_{n+1}-H_n)$$

15. $H_{2n}=2H_{n+1}P_{n+1}-3P_{2n}$

Proof:

$$H_{n+1}P_{n+1}=\left[\frac{(1+\sqrt{2})^{n+1}+(1-\sqrt{2})^{n+1}}{2}\right]\left[\frac{(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}}{2\sqrt{2}}\right]$$

$$=\frac{1}{4\sqrt{2}}\left[((1+\sqrt{2})^{n+1})^2-((1-\sqrt{2})^{n+1})^2\right]$$

$$= \frac{1}{4\sqrt{2}}\left[(1+\sqrt{2})^{2n}(3+2\sqrt{2})-(1-\sqrt{2})^{2n}(3-2\sqrt{2})\right]$$

$$= \frac{3}{2}\left[\frac{(1+\sqrt{2})^{2n}-(1-\sqrt{2})^{2n}}{2\sqrt{2}}\right]+\frac{1}{2}\left[\frac{(1+\sqrt{2})^{2n}+(1-\sqrt{2})^{2n}}{2}\right]$$

$$= \frac{1}{2}\left[3P_{2n}+H_{2n}\right]$$

$$H_{2n} = 2H_{n+1}P_{n+1} - 3P_{2n}$$

III. CONCLUSION

To conclude one may search for other patterns and their related properties.

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