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The Fractional Fourier Transform in Signal Processing

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Abstract- The Fractional Fourier Transform (FRFT.) is the generalization of the classical Fourier Transform and was introduced many years ago in mathematics literature. The original purpose of FrFT is to solve the differential equation in quantum mechanics problems in optics . Most of the applications of FrFT in optics . It depends on a parameterα and can be interpretedas rotation by an angle α in the time frequency plane or decomposition of the signal in terms of chirps. It is also used in digital domain. Because of its simple and of beautiful properties in time – frequency plane, there are many new applications waiting for signal processing.

Index Terms- Fractional Fourier Transform, signal processing and analysis, Linear Canonical Transform, Discrete FRFT

The Fractional Fourier Transform in Signal Processing

The fourth power of 3 is defined as 3^4, but $3^{3.5} = 3^{7/2} = \sqrt{3^7}$ may be defined in this manner.
The first and the second derivatives of the function f(x) as df(x)/dx and $d^2 f(x)/dx^2 = (\frac{d}{dx})^2 f(x)$
Now what is the 2.5^{th} derivative of a function? Let F(u) denote FT(Fourier Transform) of f(x).
The F.T. of the n^{th} derivative of the f(x) be given by

$(i2\pi u)^n F(u)$, for any positive integer n.

Generalizing this property by replacing n with real order a and take it as the a^{th} derivative of f(x), thus we can find $d^a f(x)/dx^a$, the a^{th} derivative of f(x) and the inverse F.T. of $(i2\pi u)^a F(u)$. Both of these will deal with the fraction of an operation performed on an entity, rather than fractions of the entity itself.

Let T be the transformation and describing T as following
T{f(x)} = F(u)
Where f and F are the two functions with variables x and u, respectively, where F is a T transform of f. thus we can also consider one new transform as

$$T^{\alpha}\{f(x)\} = F_d(u)$$

Where T^{α} , the α-order fractional T transform, and the parameter α is called the 'fractional order' and this kind of Transform is called "fractional transform", which satisfies the following conditions

<u>BOUNDARY CONDITIONS .</u>

$$T^0\{f(x)\} = f(u)$$
$$T^1\{f(x)\} = F(u).$$

<u>Additive property:</u> $T^{\beta}\{T^{\alpha}\{f(x)\}\} = T^{\beta+\alpha}\{f(x)\}$

As we know the two functions f and F are F.T. pair if

$$F(v) = \int_{-\infty}^{+\infty} f(x) \exp(-i2\pi vx)dx$$

$$\& \quad f(x) = \int_{-\infty}^{+\infty} F(v) \exp(i2\pi vx)dv$$

which reduce to operator notation as F= £{f} where £ denotes

the conventional F.T. Thus $£^2\{f(x)\} = f(-x)$; $£^4\{f(x)\} = f(x)$.

Thus the notation $£^{\alpha}$ means doing the £ for α times.

The F.T. by expressing it in terms of these Eigen functions as following

$f(x) = \sum_{n=0}^{\infty} A_n \psi_n(x)$

$A_n = \int_{-\infty}^{+\infty} \Psi_n(x)\, f(x)dx$

$£\{f(x)\} = \sum_{n=0}^{\infty} A_n e^{-in\pi/2} \psi_n(x)$

The α^{th} order Fractional Fourier Transform give the same Eigen functions as the F.T., but its Eigen value are the α^{th} power of the Eigen values of the ordinary F.T.

The Fractional operator of order α may be defined through Eigen functions of the Conventional Fourier operator as

$£^{\alpha}\{\psi_n(x)\} = e^{-i\alpha n\pi/2}\psi_n(n)$

If the operator be linear, the fractional transform of an arbitrary function may be expected as

$\{£^{\alpha}[f(x)]\}(x) = \sum_{n=0}^{\infty} A_n e^{-\alpha n\pi/2}\psi_n(x)$

Thus, we have the following essential properties:

(1) The FFT operation is Linear.

(2) The first-order Transform '£' corresponds to the Conventional F.T. £ and the zeroth-order transform $£^0$, ie. doing no transform.

(3) The fractional operator is additive £β £α = £α+β

FRCTIONAL FT. OF SOME SIGNALS

SIGNAL	Fr FT with order ∅
$\partial(t-\tau)$	$\sqrt{\frac{1-jcot\emptyset}{2}}\, .\, e^{j(\frac{\tau^2+u^2}{2})cot\emptyset}\, .\, e^{-jcosec(\emptyset\tau u)}$
$e^{-j(at^2+bt+c)}$	$\sqrt{\frac{1-jcot\emptyset}{j.2a-jcot\emptyset}}\, .\, e^{\frac{j(2acot\emptyset-1)u^2}{2(cot\emptyset-2a)}}\, .\, e^{\frac{-j(bcosec\emptyset)}{cot\emptyset-2a}}\, .\, e^{\frac{-j(b^2)}{2(cot\emptyset-2a)}-jc}$
1	$\sqrt{1+jtan\emptyset}\, .\, e^{-(\frac{ju^2tan\emptyset}{2})}$
cos(νt)	$\sqrt{1+jtan\emptyset}\, .\, e^{\frac{-j(u^2+v^2)tan\emptyset}{2}}\, .\cos(uvsec\emptyset)$
sin(νt)	$\sqrt{1+jtan\emptyset}\, .\, e^{\frac{-j(u^2+v^2)tan\emptyset}{2}}\, .\sin(uvsec\emptyset)$

We have introduced some relations between them, which are quite important because many applications are based on them. The first FRFT have several applications in the area of Optics and Signal Processing, which leads to generalization of notation of space or time and frequency domain which are central concepts of Signal processing. The generalization of F.T , Fr. F.T in a useful tool for Signal Processing. Since the flexibility of Fr. F.T is better than conventional F.T., many problems that cannot be solved well by Conventional F.T. are solved here.

It has many applications in the solutions of Differential Equation, Optical Beam Propagation and Spherical mirror resonators, Optical Diffraction theory, Optical Signal Processing, Signal Detectors and finds other relation between Fr. F.T. and other signal representation.

REFERENCES

[1] L. B. Almeida, "An introduction to the angular Fourier transform," in Proc. Con! IEEE Acoust., Speech. Signal Processing, Minneapolis, MN,Apr. 1993.

[2] L. B. Almeida, "The fractional Fourier transform and time-frequency representations," IEEE Trans. Signal Processing, vol. 42, pp.3084-3091, 1994.

[3] T. Alieva, V. Lopez, F. Aguillo-Lopez, and L. B. Almeida, "The angular Fourier transform in optical propagation problems," *J. Modern Opt., vol.*41, pp. 1037-1040, 1994.

[4] A. W. Lohmann, "Image rotation, Wigner rotation and the fractional Fourier transform," *J. Opt. Soc. Amer. A, vol.* 10, pp. 2181-2186, 1993.

[5] A. W. Lohmann and B. H. Soffer, "Relationships between the Radon-Wigner and fractional Fourier transforms," 1. *Opt. Soc. Amer. A,* vol. ll, pp, 1798-1801, 1994.

[6] D. Mendlovic and H. M. Ozaktas, "Fractional Fourier transformations and their optical implementation I," *J. Opt. Soc. Amer. A,* vol. 10, pp. 1875-1881, 1993.

[7] V. Namias, "The fractional order Fourier and its application to quantum mechanics," *J. Inst. Math. Appl.,* vol. 25, pp. 241-265, 1980.

[8] X. Xia, "On bandlimited signals with fractional Fourier transform," IEEE Signal Processing Lett., vol. 3, pp. 72-74, 1996.

[9] H. M. Ozaktas, B. Barshan, D. Mendlovic, and L. Onural, "Convolution, filtering, and multiplexing in fractional Fourier domains and their relationship to chirp and wavelet transforms," *J. Opt. Soc. Amer. A,* vol. 11, pp. 547-559, 1994.

[10] H. M. Ozaktas and D. Mendlovic, "Fractional Fourier optics," *J. Opt. Soc. Amer. A,* vol. 12, pp. 743-751, [995.

[11] "Fourier transforms of fractional order and their optical interpretation," *Opt. Commun.,* vol. 10[, pp. 163-169, 1993.

[12] A. 1. Zayed, Advances in Shannon's Sampling Theory. Boca Raton, FL: CRC, 1993.