ISSN 2250-3153

# Some Generalizations of Enestrom –Kakeya Theorem

**M. H. Gulzar**

Department of Mathematics, University of Kashmir, Srinagar 190006
E-mail: gulzarmh@gmail.com

***Abstract***- Many generalizations of the Enestrom –Kakeya Theorem are available in the literature. In this paper we give generalizations of some recent results on the subject.
**Mathematics Subject Classification :** 30C10, 30C15

***Index Terms***- Complex number, Polynomial , Zero, Enestrom –Kakeya Theorem

## I. Introduction and Statement of Results

The following result known as the Enestrom –Kakeya Theorem [3] is well-known in the theory of the distribution of zeros of polynomials:

**Theorem A:** Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n whose coefficients satisfy

$$a_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 > 0.$$

Then P(z) has all its zeros in the closed unit disk $|z| \le 1$.

In the literature there exist several generalizations and extensions of this result .Recently, Y. Choo [1] proved the following results:

**Theorem B:** Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n such that for some real number $\lambda \ne 1$,, $1 \le k \le n, a_{n-k} \ne 0,$

$$a_n \ge a_{n-1} \ge ... \ge a_{n-k+1} \ge \lambda a_{n-k} \ge a_{n-k-1} \ge ... \ge a_1 \ge a_0.$$

If $a_{n-k-1} > a_{n-k}$, then P(z) has all its zeros in the disk $|z| \le K_1$, where $K_1$ is the greatest positive root of the equation

$$K^{k+1} - \delta_1 K^k - |\gamma_1| = 0$$

and

$$\gamma_1 = \frac{(\lambda-1)a_{n-k}}{a_n}, \qquad \delta_1 = \frac{a_n + (\lambda-1)a_{n-k} - a_0 + |a_0|}{|a_n|}.$$

If $a_{n-k} > a_{n-k+1}$, then P(z) has all its zeros in the disk $|z| \le K_2$, where $K_2$ is the greatest positive root of the equation

$$K^k - \delta_2 K^{k-1} - |\gamma_2| = 0$$

and

$$\gamma_2 = \frac{(1-\lambda)a_{n-k}}{a_n}, \qquad \delta_2 = \frac{a_n + (1-\lambda)a_{n-k} - a_0 + |a_0|}{|a_n|}.$$

**Theorem C:** Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with $\operatorname{Re}(a_j)=\alpha_j$ and $\operatorname{Im}(a_j)=\beta_j$, j=0,1,2,……,n and for some real numbers $\lambda \ne 1$, $1 \le k \le n, \alpha_{n-k} \ne 0,$

$$\alpha_n \ge \alpha_{n-1} \ge ...\alpha_{n-k+1} \ge \lambda\alpha_{n-k} \ge \alpha_{n-k-1} \ge ... \ge \alpha_1 \ge \alpha_0$$

$$\beta_n \ge \beta_{n-1} \ge ...... \ge \beta_1 \ge \beta_0.$$

If $\alpha_{n-k-1} > \alpha_{n-k}$, then P(z) has all its zeros in the disk $|z| \le K_1$, where $K_1$ is the greatest positive root of the equation

$$K^{k+1} - \delta_1 K^k - |\gamma_1| = 0$$

and

$$\gamma_1 = \frac{(\lambda - 1)\alpha_{n-k}}{a_n},$$

$$\delta_1 = \frac{\alpha_n + (\lambda - 1)\alpha_{n-k} - \alpha_0 + |\alpha_0| + \beta_n - \beta_0 + |\beta_0|}{|a_n|}.$$

If $\alpha_{n-k} > \alpha_{n-k+1}$, then P(z) has all its zeros in the disk $|z| \le K_2$, where $K_2$ is the greatest positive root of the equation

$$K^k - \delta_2 K^{k-1} - |\gamma_2| = 0$$

and

$$\gamma_2 = \frac{(1 - \lambda)\alpha_{n-k}}{a_n},$$

$$\delta_2 = \frac{\alpha_n + (1 - \lambda)\alpha_{n-k} - \alpha_0 + |\alpha_0| + \beta_n - \beta_0 + |\beta_0|}{|a_n|}.$$

**Theorem D:** Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some real number $\beta$,

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2},\ j = 0,1,\ldots\ldots,n.$$

and for some $\lambda \ne 1$ and $a_{n-k} \ne 0$,

$$|a_n| \ge |a_{n-1}| \ge \ldots \ge |a_{n-k+1}| \ge \lambda |a_{n-k}| \ge |a_{n-k-1}| \ge \ldots \ge |a_1| \ge |a_0|.$$

If $|a_{n-k-1}| > |a_{n-k}|$ (i.e. $\lambda > 1$), then P(z) has all its zeros in the disk $|z| \le K_1$, where $K_1$ is the greatest positive root of the equation

$$K^{k+1} - \delta_1 K^k - |\gamma_1| = 0$$

and

$$\gamma_1 = \frac{(\lambda - 1)a_{n-k}}{a_n},$$

$$\delta_1 = \frac{\{|a_n| + (\lambda - 1)|a_{n-k}|\}(\cos\alpha + \sin\alpha) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j|}{|a_n|}.$$

If $|a_{n-k}| > |a_{n-k+1}|$ (i.e. $0 < \lambda < 1$), then P(z) has all its zeros in the disk $|z| \le K_2$, where $K_2$ is the greatest positive root of the equation

$$K^k - \delta_2 K^{k-1} - |\gamma_2| = 0$$

and

$$\gamma_2 = \frac{(1 - \lambda)a_{n-k}}{a_n},$$

$$\delta_2 = \frac{\{|a_n| + (1 - \lambda)|a_{n-k}|\}(\cos\alpha + \sin\alpha) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j|}{|a_n|}.$$

In this paper we generalize the above results with less restrictive conditions on the coefficients. In fact, we prove the following results:

**Theorem 1 :** Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n such that for some real numbers $\rho\geq 0, 0<\mu\leq 1$, $\lambda\neq 1$, , $1\leq k\leq n, a_{n-k}\neq 0,$

$$\rho+a_n\geq a_{n-1}\geq ...\geq a_{n-k+1}\geq \lambda a_{n-k}\geq a_{n-k-1}\geq ...\geq a_1\geq \mu a_0.$$

If $a_{n-k-1}>a_{n-k}$, then P(z) has all its zeros in the disk $|z|\leq K_1$, where $K_1$ is the greatest positive root of the equation

$$K^{k+1}-\delta_1 K^k-|\gamma_1|=0$$

and

$$\gamma_1=\frac{(\lambda-1)a_{n-k}}{a_n},$$

$$\delta_1=\frac{2\rho+a_n+(\lambda-1)a_{n-k}-\mu(a_0+|a_0|)+2|a_0|}{|a_n|}.$$

If $a_{n-k}>a_{n-k+1}$, then P(z) has all its zeros in the disk $|z|\leq K_2$, where $K_2$ is the greatest positive root of the equation

$$K^k-\delta_2 K^{k-1}-|\gamma_2|=0$$

and

$$\gamma_2=\frac{(1-\lambda)a_{n-k}}{a_n},$$

$$\delta_2=\frac{2\rho+a_n+(1-\lambda)a_{n-k}-\mu(a_0+|a_0|)+2|a_0|}{|a_n|}.$$

**Remark 1:** Taking $\rho=0, \mu=1,$ Theorem 1 reduces to Theorem B.
Taking $\rho=0$, Theorem 1 gives the following result:

**Corollory 1:** Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n such that for some real number $0<\mu\leq 1$, $\lambda\neq 1$, , $1\leq k\leq n, a_{n-k}\neq 0,$

$$a_n\geq a_{n-1}\geq ...\geq a_{n-k+1}\geq \lambda a_{n-k}\geq a_{n-k-1}\geq ...\geq a_1\geq \mu a_0.$$

If $a_{n-k-1}>a_{n-k}$, then P(z) has all its zeros in the disk $|z|\leq K_1$, where $K_1$ is the greatest positive root of the equation

$$K^{k+1}-\delta_1 K^k-|\gamma_1|=0$$

and

$$\gamma_1=\frac{(\lambda-1)a_{n-k}}{a_n},$$

$$\delta_1=\frac{a_n+(\lambda-1)a_{n-k}-\mu(a_0+|a_0|)+2|a_0|}{|a_n|}.$$

If $a_{n-k}>a_{n-k+1}$, then P(z) has all its zeros in the disk $|z|\leq K_2$, where $K_2$ is the greatest positive root of the equation

$$K^k-\delta_2 K^{k-1}-|\gamma_2|=0$$

and

$$\gamma_2=\frac{(1-\lambda)a_{n-k}}{a_n},$$

$$\delta_2 = \frac{a_n + (1-\lambda)a_{n-k} - \mu(a_0 + |a_0|) + 2|a_0|}{|a_n|}.$$

**Theorem 2:** Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j$ and $\text{Im}(a_j) = \beta_j$, j=0,1,2,……,n and for some real numbers $\rho \geq 0, 0 < \mu \leq 1$, $\lambda \neq 1$, $1 \leq k \leq n, \alpha_{n-k} \neq 0,$

$$\rho + \alpha_n \geq \alpha_{n-1} \geq ... \alpha_{n-k+1} \geq \lambda\alpha_{n-k} \geq \alpha_{n-k-1} \geq ... \geq \alpha_1 \geq \mu\alpha_0$$

$$\beta_n \geq \beta_{n-1} \geq ...... \geq \beta_1 \geq \beta_0.$$

If $\alpha_{n-k-1} > \alpha_{n-k}$, then P(z) has all its zeros in the disk $|z| \leq K_1$, where $K_1$ is the greatest positive root of the equation

$$K^{k+1} - \delta_1 K^k - |\gamma_1| = 0$$

and

$$\gamma_1 = \frac{(\lambda - 1)\alpha_{n-k}}{a_n},$$

$$\delta_1 = \frac{2\rho + \alpha_n + (\lambda-1)\alpha_{n-k} - \mu(\alpha_0 + |\alpha_0|) + 2|\alpha_0| + \beta_n - \beta_0 + |\beta_0|}{|a_n|}.$$

If $\alpha_{n-k} > \alpha_{n-k+1}$, then P(z) has all its zeros in the disk $|z| \leq K_2$, where $K_2$ is the greatest positive root of the equation

$$K^k - \delta_2 K^{k-1} - |\gamma_2| = 0$$

and

$$\gamma_2 = \frac{(1-\lambda)\alpha_{n-k}}{a_n},$$

$$\delta_2 = \frac{2\rho + \alpha_n + (1-\lambda)\alpha_{n-k} - \mu(\alpha_0 + |\alpha_0|) + 2|\alpha_0| + \beta_n - \beta_0 + |\beta_0|}{|a_n|}.$$

**Remark 2:** Taking $\rho = 0, \mu = 1,$ Theorem 2 reduces to Theorem C.

Taking $\rho = 0$, Theorem 2 gives the following result:

**Corollary 2:** Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j$ and $\text{Im}(a_j) = \beta_j$, j=0,1,2,……,n and for some real numbers $0 < \mu \leq 1$, $\lambda \neq 1$, $1 \leq k \leq n, \alpha_{n-k} \neq 0,$

$$\alpha_n \geq \alpha_{n-1} \geq ... \alpha_{n-k+1} \geq \lambda\alpha_{n-k} \geq \alpha_{n-k-1} \geq ... \geq \alpha_1 \geq \mu\alpha_0$$

$$\beta_n \geq \beta_{n-1} \geq ...... \geq \beta_1 \geq \beta_0.$$

If $\alpha_{n-k-1} > \alpha_{n-k}$, then P(z) has all its zeros in the disk $|z| \leq K_1$, where $K_1$ is the greatest positive root of the equation

$$K^{k+1} - \delta_1 K^k - |\gamma_1| = 0$$

and

$$\gamma_1 = \frac{(\lambda - 1)\alpha_{n-k}}{a_n},$$

$$\delta_1 = \frac{\alpha_n + (\lambda-1)\alpha_{n-k} - \mu(\alpha_0 + |\alpha_0|) + 2|\alpha_0| + \beta_n - \beta_0 + |\beta_0|}{|a_n|}.$$

If $\alpha_{n-k} > \alpha_{n-k+1}$, then P(z) has all its zeros in the disk $|z| \leq K_2$, where $K_2$ is the greatest positive root of the equation

$$K^k - \delta_2 K^{k-1} - |\gamma_2| = 0$$

and

$$\gamma_2 = \frac{(1-\lambda)\alpha_{n-k}}{a_n},$$

$$\delta_2 = \frac{\alpha_n + (1-\lambda)\alpha_{n-k} - \mu(\alpha_0 + |\alpha_0|) + 2|\alpha_0| + \beta_n - \beta_0 + |\beta_0|}{|a_n|}.$$

**Theorem 3:** Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some real number $\beta$,

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2},\ j = 0,1,......,n.$$

and for some $\lambda \ne 1$ and $a_{n-k} \ne 0$,

$$|\rho + a_n| \ge |a_{n-1}| \ge ... \ge |a_{n-k+1}| \ge \lambda|a_{n-k}| \ge |a_{n-k-1}| \ge ... \ge |a_1| \ge \mu|a_0|.$$

If $|a_{n-k-1}| > |a_{n-k}|$ (i.e. $\lambda > 1$), then P(z) has all its zeros in the disk $|z| \le K_1$, where $K_1$ is the greatest positive root of the equation

$$K^{k+1} - \delta_1 K^k - |\gamma_1| = 0$$

and

$$\gamma_1 = \frac{(\lambda-1)a_{n-k}}{a_n},$$

$$\delta_1 = \frac{[\rho + \{|\rho + a_n| + (\lambda-1)|a_{n-k}|\}(\cos\alpha + \sin\alpha) - \mu|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0| + 2\sin\alpha\sum_{j=1}^{n-1}|a_j|]}{|a_n|}.$$

If $|a_{n-k}| > |a_{n-k+1}|$ (i.e. $0 < \lambda < 1$), then P(z) has all its zeros in the disk $|z| \le K_2$, where $K_2$ is the greatest positive root of the equation

$$K^k - \delta_2 K^{k-1} - |\gamma_2| = 0$$

and

$$\gamma_2 = \frac{(1-\lambda)a_{n-k}}{a_n},$$

$$\delta_2 = \frac{[\rho + \{|\rho + a_n| + (1-\lambda)|a_{n-k}|\}(\cos\alpha + \sin\alpha) - \mu|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0| + 2\sin\alpha\sum_{j=1}^{n-1}|a_j|]}{|a_n|}$$

**Remark 3:** Taking $\rho = 0, \mu = 1,$ Theorem 4 reduces to Theorem D.

Taking $\rho = 0$, Theorem 3 gives the following result:

**Corollory3 :** Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some real number $\beta$,

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2},\ j = 0,1,......,n.$$

and for some $\lambda \ne 1$ and $a_{n-k} \ne 0$,

$$|a_n| \ge |a_{n-1}| \ge ... \ge |a_{n-k+1}| \ge \lambda|a_{n-k}| \ge |a_{n-k-1}| \ge ... \ge |a_1| \ge \mu|a_0|.$$

If $|a_{n-k-1}| > |a_{n-k}|$ (i.e. $\lambda > 1$), then P(z) has all its zeros in the disk $|z| \le K_1$, where $K_1$ is the greatest positive root of the equation

$$K^{k+1} - \delta_1 K^k - |\gamma_1| = 0$$

and

$$\gamma_1 = \frac{(\lambda - 1)a_{n-k}}{a_n},$$

$$\delta_1 = \frac{[\{|a_n| + (\lambda - 1)|a_{n-k}|\}(\cos\alpha + \sin\alpha) - \mu|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0| + 2\sin\alpha\sum_{j=1}^{n-1}|a_j|]}{|a_n|}.$$

If $|a_{n-k}| > |a_{n-k+1}|$ (i.e. $0 < \lambda < 1$), then P(z) has all its zeros in the disk $|z| \le K_2$, where $K_2$ is the greatest positive root of the equation

$$K^k - \delta_2 K^{k-1} - |\gamma_2| = 0$$

and

$$\gamma_2 = \frac{(1 - \lambda)a_{n-k}}{a_n},$$

$$\delta_2 = \frac{[\{|a_n| + (1 - \lambda)|a_{n-k}|\}(\cos\alpha + \sin\alpha) - \mu|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0| + 2\sin\alpha\sum_{j=1}^{n-1}|a_j|]}{|a_n|}.$$

## II. LEMMAS

For the proofs of the above results , we need the following result:

**Lemma 1:** Let P(z)= $\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients such that $|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}, j = 0,1,\ldots n, \text{ for}$ some real $\beta$. Then for some t>0,

$$|ta_j - a_{j-1}| \le [t|a_j| - |a_{j-1}|]\cos\alpha + [t|a_j| + |a_{j-1}|]\sin\alpha.$$

The proof of lemma 1 follows from a lemma due to Govil and Rahman [2].

## III. PROOFS OF THE THEOREMS

**Proof of Theorem 1:** We first prove that $K_1$>1 and $K_2$>1.
Let

$$f_1(K) = K^{k+1} - \delta_1 K^k - |\gamma_1|.$$

To prove $K_1$>1, it suffices to prove that $f_1(1) < 0$.

If $a_{n-k-1} > a_{n-k}$, then one of the following four cases happens:

(a) $a_{n-k+1} \ge a_{n-k-1} > a_{n-k} > 0$ and $\lambda > 1$.

(b) $a_{n-k+1} \ge a_{n-k-1} \ge 0 > a_{n-k}$ and $\lambda \le 0$.

ISSN 2250-3153

(c) $a_{n-k+1} \geq 0 \geq a_{n-k-1} > a_{n-k}$ and $\lambda < 1$.

(d) $0 \geq a_{n-k+1} \geq a_{n-k-1} > a_{n-k}$ and $0 < \lambda < 1$.

It is easy to see that $\gamma_1 > 0$ and $\delta_1 \geq 1 + \gamma_1$ for the cases (a), (b) and (c) and $\delta_1 \geq 1 + |\gamma_1|$ for the case (d). Thus $f_1(1) = 1 - \delta_1 - |\gamma_1|$ <0 and we have $K_1$>1.

Again , let

$$f_2(K) = K^k - \delta_2 K^{k-1} - |\gamma_2|.$$

If $a_{n-k} > a_{n-k+1}$, then the four possible cases to occur are the following:

(a) $a_{n-k} > a_{n-k+1} \geq a_{n-k-1} \geq 0$ and $0 \leq \lambda < 1$.

(b) $a_{n-k} > a_{n-k+1} \geq 0 \geq a_{n-k-1}$ and $\lambda \leq 1$.

(c) $a_{n-k} > 0 \geq a_{n-k+1} \geq a_{n-k-1}$ and $\lambda \leq 0$.

(d) $0 > a_{n-k} > a_{n-k+1} \geq a_{n-k-1}$ and $\lambda > 1$.

For the first three cases $\gamma_2 > 0$ and $\delta_2 \geq 1 + \gamma_2$ and for the last case $\delta_2 \geq 1 + |\gamma_2|$. Hence $f_2(1) = 1 - \delta_2 - |\gamma_2|$ <0 and we have $K_2$>1.

Now, consider the polynomial

$$\begin{aligned} F(z) &= (1-z)P(z) \\ &= (1-z)(a_n z^n + a_{n-1}z^{n-1} + ...... + a_1 z + a_0) \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + ...... + (a_{n-k+1} - a_{n-k})z^{n-k+1} + (a_{n-k} - a_{n-k-1})z^{n-k} \\ &\quad + (a_{n-k-1} - a_{n-k-2})z^{n-k-1} + ...... + (a_1 - a_0)z + a_0 \end{aligned}$$

If $a_{n-k-1} > a_{n-k}$, then $a_{n-k-1} > a_{n-k}$, and we have

$$\begin{aligned} F(z) &= -a_n z^{n+1} - (\lambda - 1)a_{n-k} z^{n-k} - \rho z^n + (\rho + a_n - a_{n-1})z^n + ...... + (a_{n-k+1} - a_{n-k})z^{n-k+1} \\ &+ (\lambda a_{n-k} - a_{n-k-1})z^{n-k} + (a_{n-k-1} - a_{n-k-2})z^{n-k-1} + ...... \\ &\quad + (a_1 - \mu a_0)z + (\mu - 1)a_0 z + a_0. \end{aligned}$$

For $|z| > 1$,

$$\begin{aligned} |F(z)| &\geq \left|a_n z^{n+1} + (\lambda-1)a_{n-k} z^{n-k}\right| - \Big| - \rho z^n + (\rho + a_n - a_{n-1})z^n + ...... + (a_{n-k+1} - a_{n-k})z^{n-k+1} \\ &\quad + (\lambda a_{n-k} - a_{n-k-1})z^{n-k} + (a_{n-k-1} - a_{n-k-2})z^{n-k-1} + ...... \\ &\quad + (a_1 - \mu a_0)z + (\mu - 1)a_0 z + a_0 \Big| \\ &= |z|^{n-k}\left|a_n z^{k+1} + (\lambda - 1)a_{n-k}\right| - |z|^n \Big| -\rho + (\rho + a_n - a_{n-1}) + ...... \\ &\quad + (a_{n-k+1} - a_{n-k})\frac{1}{z^{k-1}} + (\lambda a_{n-k} - a_{n-k-1})\frac{1}{z^k} + (a_{n-k-1} - a_{n-k-2})\frac{1}{z^{k+1}} + ...... \\ &\quad + (a_1 - \mu a_0)\frac{1}{z^{n-1}} + (\mu - 1)\frac{1}{z^{n-1}} + \frac{a_0}{z^n} \Big| \\ &> |z|^{n-k}\left|a_n z^{k+1} + (\lambda - 1)a_{n-k}\right| - |z|^n[\rho + (\rho + a_n - a_{n-1}) + ...... \\ &\quad + (a_{n-k+1} - a_{n-k}) + (\lambda a_{n-k} - a_{n-k-1}) + (a_{n-k-1} - a_{n-k-2}) + ...... \\ &\quad + (a_1 - \mu a_0) + (1-\mu)|a_0| + |a_0|] \\ &= |z|^{n-k}\left|a_n z^{k+1} + (\lambda - 1)a_{n-k}\right| - |z|^n[2\rho + a_n + (\lambda - 1)a_{n-k} - \mu(a_0 + |a_0|) \end{aligned}$$

$$+2|a_0|]$$
$$>0$$

if

$$\left|z^{k+1}+(\lambda-1)\frac{a_{n-k}}{a_n}\right|>[2\rho+a_n+(\lambda-1)a_{n-k}-\mu(a_0+|a_0|)+2|a_0|]|z|^k$$

or

$$\left|z^{k+1}+\gamma_1\right|>\delta_1|z|^k.$$

This inequality holds if

$$|z|^{k+1}-|\gamma_1|>\delta_1|z|^k.$$

Hence all the zeros of F(z) whose modulus is greater than 1 lie in the disk $|z|\le K_1$, where $K_1$ is the greatest positive root of the equation

$$K^{k+1}-\delta_1K^k-|\gamma_1|=0.$$

But the zeros of F(z) whose modulus is less than or equal to 1 are already contained in the disk $|z|\le K_1$ since $K_1>1$. Therefore, all the zeros of F(z) and hence P(z) lie in $|z|\le K_1$.

If $a_{n-k}>a_{n-k+1}$, then $a_{n-k}>a_{n-k-1}$, and we have

$$F(z)=-a_nz^{n+1}-(1-\lambda)a_{n-k}z^{n-k+1}-\rho z^n+(\rho+a_n-a_{n-1})z^n+......+(a_{n-k+1}-\lambda a_{n-k})z^{n-k+1}$$
$$+(a_{n-k}-a_{n-k-1})z^{n-k}+(a_{n-k-1}-a_{n-k-2})z^{n-k-1}+......$$
$$+(a_1-\mu a_0)z+(\mu-1)a_0z+a_0.$$

For $|z|>1$,

$$|F(z)|\ge\left|a_nz^{n+1}+(1-\lambda)z^{n-k+1}\right|-\left|-\rho z^n+(\rho+a_n-a_{n-1})z^n+......+(a_{n-k+1}-\lambda a_{n-k})z^{n-k+1}\right.$$
$$+(a_{n-k}-a_{n-k-1})z^{n-k}+(a_{n-k-1}-a_{n-k-2})z^{n-k-1}+......$$
$$\left.+(a_1-\mu a_0)z+(\mu-1)a_0z+a_0\right|$$
$$=|z|^{n-k+1}\left|a_nz^k+(1-\lambda)a_{n-k}\right|-|z|^n\left|-\rho+(\rho+a_n-a_{n-1})+......\right.$$
$$+(a_{n-k+1}-\lambda a_{n-k})\frac{1}{z^{k-1}}+(a_{n-k}-a_{n-k-1})\frac{1}{z^k}+(a_{n-k-1}-a_{n-k-2})\frac{1}{z^{k+1}}+......$$
$$\left.+(a_1-\mu a_0)\frac{1}{z^{n-1}}+(\mu-1)\frac{1}{z^{n-1}}+\frac{a_0}{z^n}\right|$$
$$>|z|^{n-k+1}\left|a_nz^k+(1-\lambda)a_{n-k}\right|-|z|^n[\rho+(\rho+a_n-a_{n-1})+......$$
$$+(a_{n-k+1}-\lambda a_{n-k})+(a_{n-k}-a_{n-k-1})+(a_{n-k-1}-a_{n-k-2})+......$$
$$+(a_1-\mu a_0)+(1-\mu)|a_0|+|a_0|]$$

$$=|z|^{n-k+1}\left|a_nz^k+(1-\lambda)a_{n-k}\right|-|z|^n[2\rho+a_n+(1-\lambda)a_{n-k}-\mu(a_0+|a_0|)$$
$$+2|a_0|]$$
$$>0$$

if

$$\left|z^k+(1-\lambda)\frac{a_{n-k}}{a_n}\right|>[2\rho+a_n+(1-\lambda)a_{n-k}-\mu(a_0+|a_0|)+2|a_0|]|z|^{k-1}$$

or

ISSN 2250-3153

$$\left|z^k+\gamma_2\right|>\delta_2|z|^{k-1}.$$

This inequality holds if

$$|z|^k-|\gamma_2|>\delta_2|z|^{k-1}.$$

Hence all the zeros of F(z) whose modulus is greater than 1lie the disk $|z|\le K_2$, where $K_2$ is the greatest positive root of the equation

$$K^k-\delta_2K^{k-1}-|\gamma_2|=0.$$

But those zeros of F(z) whose modulus is less than or equal to 1 are already contained in the disk $|z|\le K_2$ since $K_2>1$. Therefore, all the zeros of F(z) and hence P(z) lie in $|z|\le K_2$.

That proves Theorem 1.

**Proof of Theorem 2:** Consider the polynomial

$$\begin{aligned}F(z)&=(1-z)P(z)\\&=(1-z)(a_nz^n+a_{n-1}z^{n-1}+......+a_1z+a_0)\\&=-a_nz^{n+1}+(a_n-a_{n-1})z^n+......+(a_{n-k+1}-a_{n-k})z^{n-k+1}+(a_{n-k}-a_{n-k-1})z^{n-k}\\&\quad+(a_{n-k-1}-a_{n-k-2})z^{n-k-1}+......+(a_1-a_0)z+a_0\\&=-a_nz^{n+1}+(\alpha_n-\alpha_{n-1})z^n+......+(\alpha_{n-k+1}-\alpha_{n-k})z^{n-k+1}\\&\quad+(\alpha_{n-k}-\alpha_{n-k-1})z^{n-k}+(\alpha_{n-k-1}-\alpha_{n-k-2})z^{n-k-1}+......+(\alpha_1-\mu\alpha_0)z\\&\quad+(\mu-1)\alpha_0z+\alpha_0+i\sum_{j=1}^{n}(\beta_j-\beta_{j-1})z^j+i\beta_0\end{aligned}$$

If $\alpha_{n-k-1}>\alpha_{n-k}$, then $\alpha_{n-k-1}>\alpha_{n-k}$, and we have

$$\begin{aligned}F(z)&=-a_nz^{n+1}-(\lambda-1)\alpha_{n-k}z^{n-k}-\rho z^n+(\rho+\alpha_n-\alpha_{n-1})z^n+......\\&\quad+(\alpha_{n-k+1}-\alpha_{n-k})z^{n-k+1}+(\lambda\alpha_{n-k}-\alpha_{n-k-1})z^{n-k}+(\alpha_{n-k-1}-\alpha_{n-k-2})z^{n-k-1}+......\\&\quad+(\alpha_1-\mu\alpha_0)z+(\mu-1)\alpha_0z+\alpha_0+i\sum_{j=1}^{n}(\beta_j-\beta_{j-1})z^j+i\beta_0.\end{aligned}$$

For $|z|>1$,

$$\begin{aligned}|F(z)|&\ge\left|a_nz^{n+1}+(\lambda-1)\alpha_{n-k}z^{n-k}\right|-|z|^n[\rho+(\rho+\alpha_n-\alpha_{n-1})+......+(\alpha_{n-k+1}-\alpha_{n-k})\frac{1}{|z|^{k-1}}\\&\quad+(\lambda\alpha_{n-k}-\alpha_{n-k-1})\frac{1}{|z|^k}+(\alpha_{n-k-1}-\alpha_{n-k-2})\frac{1}{|z|^{k+1}}+......+(\alpha_1-\mu\alpha_0)\frac{1}{|z|^{n-1}}\\&\quad+\frac{(1-\mu)|\alpha_0|}{|z|^{n-1}}+\frac{|\alpha_0|}{|z|^n}]-|z|^n[(\beta_n-\beta_{n-1})+......+(\beta_1-\beta_0)\frac{1}{|z|^{n-1}}+\frac{|\beta_0|}{|z|^n}]\\&>\left|a_nz^{n+1}+(\lambda-1)\alpha_{n-k}z^{n-k}\right|-|z|^n[2\rho+\alpha_n+(\lambda-1)\alpha_{n-k}-\mu(\alpha_0+|\alpha_0|)\\&\quad+2|\alpha_0|+\beta_n-\beta_0+|\beta_0|]\\&>0\end{aligned}$$

if

$$\left|z^{k+1}+\gamma_1\right|>\delta_1|z|^k.$$

But this inequality holds if

$$|z|^{k+1}-|\gamma_1|>\delta_1|z|^k.$$

Hence, it follows that all the zeros of F(z) whose modulus is greater than 1 lie in the disk $|z|\le K_1$, where $K_1$ is the greatest positive root of the equation

$$K^{k+1}-\delta_1 K^k-|\gamma_1|=0.$$

As in the proof of Theorem 1, it can be shown that $K_1>1$, so that the zeros of F(z) whose modulus is less than or equal to 1 are already contained in the disk $|z|\le K_1$. Therefore, all the zeros of F(z) and hence P(z) lie in $|z|\le K_1$.

Now,, suppose that $\alpha_{n-k}>\alpha_{n-k+1}$, then $\alpha_{n-k}>\alpha_{n-k-1}$, and we have

$$\begin{aligned}F(z)=&-a_nz^{n+1}-(1-\lambda)\alpha_{n-k}z^{n-k+1}-\rho z^n+(\rho+\alpha_n-\alpha_{n-1})z^n+\ldots\ldots\\&+(\alpha_{n-k+1}-\lambda\alpha_{n-k})z^{n-k+1}+(\alpha_{n-k}-\alpha_{n-k-1})z^{n-k}\\&+(\alpha_{n-k-1}-\alpha_{n-k-2})z^{n-k-1}+\ldots\ldots+(\alpha_1-\mu\alpha_0)z\\&+(\mu-1)\alpha_0z+\alpha_0+i\sum_{j=1}^{n}(\beta_j-\beta_{j-1})z^j+i\beta_0.\end{aligned}$$

For $|z|>1$,

$$\begin{aligned}|F(z)|\ge&\,|a_nz^{n+1}+(1-\lambda)\alpha_{n-k}z^{n-k+1}|-|z|^n[\rho+(\rho+\alpha_n-\alpha_{n-1})+\ldots\ldots+(\alpha_{n-k+1}-\lambda\alpha_{n-k})\frac{1}{|z|^{k-1}}\\&+(\alpha_{n-k}-\alpha_{n-k-1})\frac{1}{|z|^k}+(\alpha_{n-k-1}-\alpha_{n-k-2})\frac{1}{|z|^{k+1}}+\ldots\ldots+(\alpha_1-\mu\alpha_0)\frac{1}{|z|^{n-1}}\\&+\frac{(1-\mu)|\alpha_0|}{|z|^{n-1}}+\frac{|\alpha_0|}{|z|^n}]-|z|^n[(\beta_n-\beta_{n-1})+\ldots\ldots+(\beta_1-\beta_0)\frac{1}{|z|^{n-1}}+\frac{|\beta_0|}{|z|^n}]\\>&\,|a_nz^{n+1}+(1-\lambda)\alpha_{n-k}z^{n-k+1}|-|z|^n[2\rho+\alpha_n+(1-\lambda)\alpha_{n-k}-\mu(\alpha_0+|\alpha_0|)\\&+2|\alpha_0|+\beta_n-\beta_0+|\beta_0|]\\>&\,0\end{aligned}$$

if

$$|z^k+\gamma_2|>\delta_2|z|^{k-1}.$$

But this inequality holds if

$$|z|^k-|\gamma_2|>\delta_2|z|^{k-1}.$$

Hence, it follows that all the zeros of F(z) whose modulus is greater than 1 lie in the disk $|z|\le K_2$, where $K_2$ is the greatest positive root of the equation

$$K^k-\delta_2K^{k-1}-|\gamma_1|=0.$$

As in the proof of Theorem 1, it can be shown that $K_2>1$.Thus the zeros of F(z) whose modulus is less than or equal to 1 are already contained in the disk $|z|\le K_2$. Therefore, all the zeros of F(z) and hence P(z) lie in $|z|\le K_2$ and the proof of Theorem 2 is complete.

**Proof of Theorem 3:** Consider the polynomial

$$\begin{aligned}F(z)&=(1-z)P(z)\\&=(1-z)(a_nz^n+a_{n-1}z^{n-1}+\ldots\ldots+a_1z+a_0)\\&=-a_nz^{n+1}+(a_n-a_{n-1})z^n+\ldots\ldots+(a_{n-k+1}-a_{n-k})z^{n-k+1}+(a_{n-k}-a_{n-k-1})z^{n-k}\end{aligned}$$

$$+(a_{n-k-1}-a_{n-k-2})z^{n-k-1}+......+(a_1-a_0)z+a_0\,.$$

If $|a_{n-k-1}|>|a_{n-k}|$, then $|a_{n-k+1}|>|a_{n-k}|$ , $\lambda>1$ and we have,

F(z) $=-a_nz^{n+1}-(\lambda-1)a_{n-k}-\rho z^n+(\rho+a_n-a_{n-1})z^n+......+(a_{n-k+1}-a_{n-k})z^{n-k+1}$

$$+(\lambda a_{n-k}-a_{n-k-1})z^{n-k}+(a_{n-k-1}-a_{n-k-2})z^{n-k-1}+......$$

$$+(a_1-\mu a_0)z+(\mu-1)a_0z+a_0.$$

For $|z|>1$, by using Lemma1,

$$|F(z)|\ge|a_nz^{n+1}+(\lambda-1)a_{n-k}z^{n-k}|-|-\rho z^n+(\rho+a_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+......+$$
$$(a_{n-k+1}-a_{n-k})z^{n-k+1}+(\lambda a_{n-k}-a_{n-k-1})z^{n-k}+(a_{n-k-1}-a_{n-k-2})z^{n-k-1}$$
$$+......+(a_1-\mu a_0)z+(\mu-1)a_0z+a_0\ |$$

$$\ge|a_nz^{n+1}+(\lambda-1)a_{n-k}z^{n-k}|-|z|^n[\rho+|\rho+a_n-a_{n-1}|+......+\frac{|a_{n-k+1}-a_{n-k}|}{|z|^{k-1}}$$
$$+\frac{|\lambda a_{n-k}-a_{n-k-1}|}{|z|^k}+......+\frac{|a_1-\mu a_0|}{|z|^{n-1}}+\frac{(1-\mu)|a_0|}{|z|^{n-1}}+\frac{|a_0|}{|z|^n}]$$

$$>|a_nz^{n+1}+(\lambda-1)a_{n-k}z^{n-k}|-|z|^n[\rho+|\rho+a_n-a_{n-1}|+......+|a_{n-k+1}-a_{n-k}|$$
$$+|\lambda a_{n-k}-a_{n-k-1}|+......+|a_1-\mu a_0|+(1-\mu)|a_0|+|a_0|]$$

$$\ge|a_nz^{n+1}+(\lambda-1)a_{n-k}z^{n-k}|-|z|^n[\rho+\{|\rho+a_n|+(\lambda-1)|a_{n-k}|\}(\cos\alpha+\sin\alpha)$$
$$-\mu|a_0|(\cos\alpha-\sin\alpha+1)+2|a_0|+2\sin\alpha\sum_{j=0,\,j\ne n-k}^{n-1}|a_j|]$$

$$>0$$

if

$$|z^{k+1}+\gamma_1|>\delta_1|z|^k\,.$$

This inequality holds if

$$|z|^{k+1}-|\gamma_1|>\delta_1|z|^k$$

and thus all the zeros of F(z )and hence P(z) with modulus greater than 1 lie in the disk $|z|\le K_1$, where $K_1$ is the greatest positive root of the equation

$$K^{k+1}-\delta_1K^k-|\gamma_1|=0\,.$$

It is easy to see that $K_1>1$ and all the zeros of P(z) with modulus less than or equal to 1 are already contained in $|z|\le K_1$.

Again , If $|a_{n-k}|>|a_{n-k+1}|$, then $|a_{n-k}|>|a_{n-k-1}|$ , $\lambda<1$ and we have,

F(z) $=-a_nz^{n+1}-(1-\lambda)a_{n-k}z^{n-k+1}-\rho z^n+(\rho+a_n-a_{n-1})z^n+......$

$$+(a_{n-k+1}-\lambda a_{n-k})z^{n-k+1}+(a_{n-k}-a_{n-k-1})z^{n-k}+(a_{n-k-1}-a_{n-k-2})z^{n-k-1}+......$$

$$+(a_1-\mu a_0)z+(\mu-1)a_0z+a_0.$$

For $|z|>1$, by using Lemma1,

$$|F(z)|\ge|a_nz^{n+1}+(1-\lambda)a_{n-k}z^{n-k+1}|-|-\rho z^n+(\rho+a_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+......+$$

$$(a_{n-k+1}-\lambda a_{n-k})z^{n-k+1}+(a_{n-k}-a_{n-k-1})z^{n-k}+(a_{n-k-1}-a_{n-k-2})z^{n-k-1}$$
$$+......+(a_1-\mu a_0)z+(\mu-1)a_0z+a_0 \;|$$
$$\geq \left|a_nz^{n+1}+(1-\lambda)a_{n-k}z^{n-k+1}\right|-|z|^n[\rho+|\rho+a_n-a_{n-1}|+......+\frac{|a_{n-k+1}-\lambda a_{n-k}|}{|z|^{k-1}}$$
$$+\frac{|a_{n-k}-a_{n-k-1}|}{|z|^k}+......+\frac{|a_1-\mu a_0|}{|z|^{n-1}}+\frac{(1-\mu)|a_0|}{|z|^{n-1}}+\frac{|a_0|}{|z|^n}]$$
$$>\left|a_nz^{n+1}+(1-\lambda)a_{n-k}z^{n-k+1}\right|-|z|^n[\rho+|\rho+a_n-a_{n-1}|+......+|a_{n-k+1}-\lambda a_{n-k}|$$
$$+|a_{n-k}-a_{n-k-1}|+......+|a_1-\mu a_0|+(1-\mu)|a_0|+|a_0|]$$
$$\geq \left|a_nz^{n+1}+(1-\lambda)a_{n-k}z^{n-k+1}\right|-|z|^n[\rho+\{|\rho+a_n|+(1-\lambda)|a_{n-k}|\}(\cos\alpha+\sin\alpha)$$
$$-\mu|a_0|(\cos\alpha-\sin\alpha+1)+2|a_0|+2\sin\alpha\sum_{j=0,\,j\neq n-k}^{n-1}|a_j|]$$
$$>0$$

if

$$\left|z^k+\gamma_2\right|>\delta_2|z|^{k-1}.$$

This inequality holds if

$$|z|^k-|\gamma_2|>\delta_2|z|^{k-1}$$

and thus all the zeros of F(z )and hence P(z) with modulus greater than 1 lie in the disk $|z|\leq K_2$, where $K_2$ is the greatest positive root of the equation

$$K^k-\delta_2K^{k-1}-|\gamma_2|=0.$$

It is easy to see that $K_2>1$ and all the zeros of P(z) with modulus less than or equal to 1 are already contained in $|z|\leq K_2$. That proves Theorem 3.

## References

[1] Y. Choo, Further Generalizations of Enestrom-Kakeya Theorem, Int. Journal of Math. Analysis. Vol. 5, 2011,no. 20, 983-995.
[2] N. K. Govil and Q. I. Rahman, On the Enestrom-Kakeya Theorem, Tohoku Math. J.,20(1968) , 126-136.
[3] M. Marden , Geometry of Polynomials, IInd Ed.Math. Surveys, No. 3, Amer. Math. Soc. Providence,R. I,1996.