An Lower Estimate for the Number of Level Crossing of a Random Algebraic Curve when the Coefficients follow Semi-Stable Distribution

N.N.Nayak[1] and N.K.Sahoo[2]

[1]Ex-Prof in Mathematics, Orissa University of Agriculture & Technology, Bhubaneswar-3, India, Plot No.1242/2171, Siripur Nuasshi
[2]Faculty of Science and Technology, The ICFAI University Tripura, Kamalghat, Sadar, 799210, India

Abstract- Let $\xi_1(\omega), \xi_2(\omega), \xi_3(\omega) \dots\dots\dots\dots\dots .\xi_n(\omega)$ be a sequence of mutually independent, identically distributed random variables following semi-stable distribution with characteristic function $exp(-(C + coslog \mid t \mid) \mid t \mid^{\alpha}),\quad C > 1,\ 1 < \alpha < 2$. In this work, we obtain the lower bound of the number of real zeros of the random algebraic equation$\sum_{r=0}^{n} \xi_n(\omega)x^r$. $N_n(\omega)$ denote the number of real roots must $> \left({}^{\mu logn}/_{loglogn} \right)\ for\ all\ n > n_0$, except for a set of measure at most $\frac{\mu' loglogn}{logn}$.

Index Terms- Random variables, Joint distribution, Characteristic function, Semi-stable distribution, Multiple roots, Random algebraic equations.

I. Introduction

Let $N_n(\omega)$ be the number of real zeros of the nth degree polynomial $\sum_{r=0}^{n} \xi_r x^r$ where the coefficients $\xi_r's$ are identically distributed independent random variables. Ibragimov and Maslova first studied the average number of real roots when the coefficients belong to the domain of attraction of normal law and established that $EN_n \sim \frac{1}{\pi} logn$. Samal and Mishra [6], [7], [8] have studied the lower bound for the number of real roots of the equation$\sum_{r=0}^{n} \xi_r x^r = 0$, where the coefficients $\xi_r's$ are symmetric stable random variables with infinite variance i.e, with characteristic function $exp(-C \mid t \mid^{\alpha}), C > 0\ and\ 1 < \alpha < 2.$ In this paper we have studied the lower bound for the number of real roots where $\xi_r's$ follow semi-stable distribution. Thus we have extended the results of Samal and Mishra to wider class of random variables. Thus our result is a generalization of earlier works in this respect

Theorem -1
Let $f(x) = \sum_{r=0}^{n} \xi_n(\omega)x^r$ be an algebraic polynomial, where $\xi_r's$ are random variables following semi-stable distribution with common characteristics function $\varphi(t) = exp(-(C + coslog \mid t \mid) \mid t \mid^{\alpha})$ for constant $C > 1,\quad 1 < \alpha < 2$.
Then there exist $n_0 \in N$ such that for $n > n_0$, N_n the number of level crossing satisfy of the random algebraic polynomial$f(x) = 0$.
$N_n(\omega) > \left({}^{\mu logn}/_{loglogn} \right)\ for\ all\ n > n_0$, out side a set of measure at most $\frac{\mu' loglogn}{logn}$.

II. Preliminaries

We need the following lemmas for the proof of the theorem.
Choose constants $A\ and\ B$ such that

$0 < B < 1\,, A > 1$ and let $\lambda = logn$

Let M be the integer determined by $M = \left[\frac{2^{\alpha+1} Ae\lambda^{\alpha}}{B}\right] + 1$

Let k be an integer determined by
$$M^{2k} \le n \le M^{2k+2} \qquad (2.2.1)$$

Since the number of roots in $(-1,1)$ is twice in $[0,1]$ it will be four times in $(-\infty, \infty)$ as transformation $x = \frac{1}{t}$ will reduce $(0, \infty)\ to\ [0,1]$. There fore it will be sufficient to consider the points x_m given by

$$x_m = \left(1 - \frac{1}{M^{2m}}\right)^{\frac{1}{\alpha}} \qquad (2.2.2)$$

Where $m \in S = \left\{\left[\frac{k}{2}\right]+1, \left[\frac{k}{2}\right]+2, \left[\frac{k}{2}\right]+3, \dots \dots . k\right\}$

There are $\left[\frac{k}{2}\right]$ points if k is even and $\left(\frac{k+1}{2}\right)$ points if k is odd.

Clearly when $m \to \infty, k \to \infty\ and\ x_m \to 1$ and

When $m \to 0 (i.e\ k \to 0), x_m \to 0$. So that x_m covers all the points in [0,1].

Ste $f(x_m) = A_m + R_m$, where $A_m = \sum_1 \xi_r(\omega) {x_m}^r = \sum_{r=M^{2m-1}+1}^{M^{2m+1}} \xi_r(\omega) {x_m}^r$

And $R_m = \sum_2 \xi_r(\omega) {x_m}^r + \sum_3 \xi_r(\omega) {x_m}^r$

$R_m = \sum_{r=0}^{M^{2m-1}} \xi_r(\omega) {x_m}^r + \sum_{r=M^{2m+1}+1}^{n} \xi_r(\omega) {x_m}^r$

r ranging from $M^{2m-1} + 1\ to\ M^{2m+1}\ in\ \sum_1 \xi_r(\omega) {x_m}^r$,

from 0 $to\ M^{2m-1} in\ \sum_2 \xi_r(\omega) {x_m}^r$ and from $M^{2m+1} + 1\ to\ n\ in\ \sum_3 \xi_r(\omega) {x_m}^r$ respectively.

Obviously A_m, A_{m+1} are independent random variables. We defined normalizing constants V_m by

${V_m}^\alpha = \sum_1 {x_m}^{\alpha r} = \sum_{r=M^{2m-1}+1}^{M^{2m+1}} {x_m}^{\alpha r}$.

Lemma 2.1
If $\varphi(t)$ is characteristic function of a random variables $\xi(\omega)$ which follow semi-normal distribution, then

$P(\omega: |\xi(\omega)| > \epsilon) \leq \frac{c+1}{(\alpha+1)\epsilon^\alpha}$

Proof of the lemma
By Loeve [3] , we have

$P\{\omega: |\xi(\omega)| > \epsilon\} < 7\epsilon \int_0^{\frac{1}{\epsilon}} \left(1 - \varphi(t)\right) dt = 7\epsilon(c+1) \int_0^{\frac{1}{\epsilon}} t^\alpha\, dt$

$\therefore\ P\{\omega: |\xi(\omega)| > \epsilon\} \leq \frac{7(c+1)}{(\alpha+1)\epsilon^\alpha}$

Lemma 2.2
$|\sum_{r=M^{2m+1}+1}^{n} \xi_r(\omega) {x_m}^r| < \frac{1}{2} V_m$, except for a set of measure at most $\frac{2^\alpha}{\alpha+1} \frac{Ae}{B} (c+1) e^{-M}$

Proof of the lemma
The characteristic function of $\sum_3 \xi_r(\omega) {x_m}^r$ is $exp\left\{-(c + coslog|t|)|t|^\alpha . \sum_{r=M^{2m+1}+1}^{n} {x_m}^{\alpha r}\right\}$ $for\ 1 < \alpha < 2$

Hence by lemma (2.1), we have

$P_1 = P\left\{| \sum_{r=M^{2m+1}+1}^{n} \xi_r(\omega) {x_m}^r | \geq \frac{1}{2} V_m\right\}$

$\leq \frac{(c+1)}{(\alpha+1)\left(\frac{V_m}{2}\right)^\alpha} . \sum_{r=M^{2m+1}+1}^{n} {x_m}^{\alpha r}$ (2.2.3)

$\therefore \sum_{r=M^{2m+1}+1}^{n} {x_m}^{\alpha r} < M^{2m} e^{-M}$

$\Rightarrow P_1 = P\left\{| \sum_{r=M^{2m+1}+1}^{n} \xi_r(\omega) {x_m}^r | \geq \frac{1}{2} V_m\right\} < \frac{(C+1)2^\alpha}{\alpha+1} \frac{Ae}{B} e^{-M}$

Lemma 2.3

$|\sum_{r=0}^{M^{2m-1}} \xi_r(\omega){x_m}^r| < \lambda\left(\sum_{r=0}^{M^{2m-1}} {x_m}^{\alpha r}\right)^{\frac{1}{\alpha}}$,

except for a set of measure at most $\frac{(C+1)}{(\alpha+1)}\frac{1}{\lambda^\alpha}$.

Proof of the lemma

The characteristic function of $\sum_{r=0}^{M^{2m-1}} \xi_r(\omega){x_m}^r$ is $exp\{-(c + coslog|t|)|t|^\alpha . \sum_{r=0}^{M^{2m-1}} {x_m}^{\alpha r}\}$ $\quad for\ 1 < \alpha < 2$

Let $P_2 = P\left\{|\sum_{r=0}^{M^{2m-1}} \xi_r(\omega){x_m}^r| \geq \lambda\left(\sum_{r=0}^{M^{2m-1}} {x_m}^{\alpha r}\right)^{\frac{1}{\alpha}}\right\}$

Let ${D^\alpha}_m = \sum_{r=0}^{M^{2m-1}} {x_m}^{\alpha r}$ and the characteristic function of $\sum_{r=0}^{M^{2m-1}} \xi_r(\omega){x_m}^r$ be $\varphi(t)$

So $P_2 = P\left\{|\sum_{r=0}^{M^{2m-1}} \xi_r(\omega){x_m}^r| \geq \lambda\left(\sum_{r=0}^{M^{2m-1}} {x_m}^{\alpha r}\right)^{\frac{1}{\alpha}}\right\} \leq \frac{(C+1)}{(\alpha+1)}\frac{1}{\lambda^\alpha}$

So $P_2 \leq \frac{(C+1)}{(\alpha+1)}\frac{1}{\lambda^\alpha}$

Now by using Lemmas $(2.2\ )$ and $(2.3\ )$, we have for any given m

$|R_m| < \frac{1}{2}V_m + \lambda D_m$, except for a set of measure at most

$\frac{(C+1)2^\alpha}{\alpha+1}\frac{Ae}{B}e^{-M} + \frac{(C+1)}{(\alpha+1)}\frac{1}{\lambda^\alpha}$, but $\lambda\left(\sum_{r=0}^{M^{2m-1}} {x_m}^{\alpha r}\right)^{\frac{1}{\alpha}} < \lambda V_m\left(\frac{2Ae}{MB}\right)^{\frac{1}{\alpha}}$

Hence $R_m < \frac{1}{2}V_m + \frac{1}{2}V_m = V_m$, for $m \in S$ except for a set of measure less than equal to

$$= \frac{(C+1)}{(\alpha+1)}\left(2^\alpha\frac{Ae}{B}e^{-M} + \frac{1}{\lambda^\alpha}\right) \qquad (2.2.4)$$

We define events $E_m\ and\ F_m$ by

$E_m = \{\omega: A_{2m}(\omega) > V_{2m},\ A_{2m+1}(\omega) < -V_{2m+1}\},\quad F_m = \{\omega: A_{2m}(\omega) < -V_{2m},\ A_{2m+1}(\omega) > V_{2m+1}\}$

Lemma 2.4

Union of events $E_m\ and\ F_m$ defined by

$$E_m = \{\omega: A_{2m}(\omega) > V_{2m}\} \cap \{\omega: A_{2m+1}(\omega) < -V_{2m+1}\}$$
$$F_m = \{\omega: A_{2m}(\omega) < -V_{2m}\} \cap \{\omega:\ A_{2m+1}(\omega) > V_{2m+1}\}$$

occur with positive probability.

Proof:

Let $g_m(u)\ and\ G_m(x)$ denote respectively characteristic function and distribution function of the random variables $\left(\frac{A_m(\omega)}{V_m}\right)$. Then $g_m(u) = exp\left\{-(c + coslog|t|)|t|^\alpha \frac{1}{{V_m}^\alpha}\sum_{M^{2m-1}+1}^{M^{2m+1}} {x_m}^{\alpha r} L\left({x_m}^r\frac{u}{V_m}\right)\right\}$.

obviously $V_m \to \infty\ as\ m \to \infty$ and as such

${x_m}^\alpha \frac{u}{V_m} \to 0\ \ and\ \ {x_m}^r\frac{\theta}{V_m} \to 0\ as\ m \to \infty$.

We use lemma 6.3 here taking

$\lim_{u\to 0} C(x) = C$ and get

ISSN 2250-3153

$$\frac{L\left(\left|\frac{x_m{}^r t}{V_m}\right|\right)}{L\left(\left|\frac{x_m{}^r \theta}{V_m}\right|\right)} = \frac{C\left(\left|\frac{x_m{}^r t}{V_m}\right|\right) exp\left\{-\int_a^{\left(\left|\frac{x_m{}^r t}{V_m}\right|\right)} -\left(c+coslog\left|\frac{\epsilon(u)}{u}\right|\right)\left|\frac{\epsilon(u)}{u}\right|^{\alpha} du\right\}}{C\left(\left|\frac{x_m{}^r \theta}{V_m}\right|\right) exp\left\{-\int_a^{\left(\left|\frac{x_m{}^r \theta}{V_m}\right|\right)} -\left(c+coslog\left|\frac{\epsilon(u)}{u}\right|\right)\left|\frac{\epsilon(u)}{u}\right|^{\alpha} du\right\}}.$$

Since $\lim_{u\to 0} C(x) = C$

and

since $\frac{x_m{}^r t}{V_m} \to 0 \quad and \quad \frac{x_m{}^r \theta}{V_m} \to 0 \; as \; m \to \infty$, we have

$$C\left(\left|\frac{x_m{}^r t}{V_m}\right|\right) = \left(1 + o(1)\right)C \; as \; m \to \infty,$$

And

$$C\left(\left|\frac{x_m{}^r \theta}{V_m}\right|\right) = \left(1 + o(1)\right)C \; as \; m \to \infty.$$

Also since $\lim_{u\to 0} \epsilon(u) = 0,\ \epsilon > 0$ there exists $t_o > 0$ such that for $u < t_o$, $|\epsilon(u)| < \epsilon$

so, $\epsilon(u) = \left(1 + o(1)\right)\epsilon \;\; as \; u \to 0.$

we have

$$\frac{L\left(\left|\frac{x_m{}^r t}{V_m}\right|\right)}{L\left(\left|\frac{x_m{}^r \theta}{V_m}\right|\right)} = \frac{C\left(\left|\frac{x_m{}^r t}{V_m}\right|\right) exp\left\{-\int_a^{\left(\left|\frac{x_m{}^r t}{V_m}\right|\right)} -\left(c+coslog\left|\frac{\epsilon(u)}{u}\right|\right)\left|\frac{\epsilon(u)}{u}\right|^{\alpha} du\right\}}{C\left(\left|\frac{x_m{}^r \theta}{V_m}\right|\right) exp\left\{-\int_a^{\left(\left|\frac{x_m{}^r \theta}{V_m}\right|\right)} -\left(c+coslog\left|\frac{\epsilon(u)}{u}\right|\right)\left|\frac{\epsilon(u)}{u}\right|^{\alpha} du\right\}}$$

$$= \frac{C\left(\left|\frac{x_m{}^r t}{V_m}\right|\right)}{C\left(\left|\frac{x_m{}^r \theta}{V_m}\right|\right)} exp\left\{-\int_{\left(\left|\frac{x_m{}^r \theta}{V_m}\right|\right)}^{\left(\left|\frac{x_m{}^r t}{V_m}\right|\right)} -\left(c + coslog\left|\frac{\epsilon(u)}{u}\right|\right)\left|\frac{\epsilon(u)}{u}\right|^{\alpha} du\right\}$$

$$= \frac{(1+o(1))C}{(1+o(1))C} exp\left\{-\int_{\left(\left|\frac{x_m{}^r \theta}{V_m}\right|\right)}^{\left(\left|\frac{x_m{}^r t}{V_m}\right|\right)} \frac{\epsilon}{u}\left(1 + o(1)\right) du\right\} \quad as \; u \to 0$$

$$= \epsilon exp\left\{-\int_{\left(\left|\frac{x_m{}^r \theta}{V_m}\right|\right)}^{\left(\left|\frac{x_m{}^r t}{V_m}\right|\right)} \frac{1}{u} du\right\}\left(1 + o(1)\right)$$

$$= \left(1 + o(1)\right)e^{log\left|\frac{\theta}{t}\right|^{\epsilon}} = \left|\frac{\theta}{t}\right|^{o(1)} \;\; as \; u \to 0 \; making \; \epsilon \to 0$$

which implies

$$L\left(\left|\frac{x_m{}^r t}{V_m}\right|\right) = \left|\frac{\theta}{t}\right|^{o(1)} L\left(\left|\frac{x_m{}^r \theta}{V_m}\right|\right) \;\; as \; u \to 0\,.$$

Hence

$$g_m(u) \to exp\left\{-(c + coslog|u|)|u|^{\alpha}\frac{1}{V_m{}^{\alpha}}\left|\frac{\theta}{t}\right|^{o(1)} \sum_{M^{2m-1}+1}^{M^{2m+1}} x_m{}^{\alpha r} L\left(\frac{x_m{}^r \theta}{V_m}\right)\right\}$$

ISSN 2250-3153

$= exp\left\{-(c + coslog|u|)|u|^{\alpha-o(1)}\theta^{o(1)}\frac{1}{V_m{}^{\alpha}}\sum_{M^{2m-1}+1}^{M^{2m+1}} x_m{}^{\alpha r} L\left(\frac{x_m{}^r\theta}{V_m}\right)\right\}$

$= exp\{-(c + coslog|u|)|u|^{\alpha-o(1)}\theta^{o(1)}\}$

(by definition of V_m).

So as $m \to \infty, g_m(u) \to exp\{-(c + coslog|u|)|u|^{\alpha}\}$ in any bounded interval of u − values. Since $exp\{-|u|^{\alpha}\}$ is continuous for all u, $G_m(x)$ will converge to a distribution $F(x)$ (say) corresponding to the characteristic function $g_m(u)$ (cf Gnedenko and Kolmagorov [17] p.83).
so
$\sup_x|G_m(x) - F(x)| = o(1)$ (2.2.5)

So for $\epsilon > 0, |G_{2m}(-1) - F(-1)| < \epsilon$

$\Rightarrow F(-1) - \epsilon < G_{2m}(-1) < F(-1) + \epsilon$

and $|G_{2m+1}(-1) - F(-1)| < \epsilon$

implies $F(-1) - \epsilon < G_{2m+1}(-1) < F(-1) + \epsilon$.

Then $P(A_{2m} < -V_{2m}) = P\left(\frac{A_{2m}}{V_{2m}} < -1\right)$

$G_{2m}(-1) > F(-1) - \epsilon \ as\ m \to \infty$

and similarly
$P(A_{2m+1} < -V_{2m+1}) > F(-1) - \epsilon$

also
$|G_{2m}(1) - F(1)| < \epsilon$

$\Rightarrow F(1) - \epsilon < G_{2m}(1) < F(1) + \epsilon$

$-(F(1) - \epsilon) > -G_{2m}(1)$

$1-(F(1) - \epsilon) > 1 - G_{2m}(1)$

or
$1 - G_{2m}(1) < 1 - F(1) + \epsilon$ (2.2.6)

So,
$P(A_{2m} > V_{2m} = 1 - P(A_{2m} < V_{2m})$.

$= 1 - G_{2m}(1)$
$> 1 - F(1) - \epsilon$ (2.2.7)

Similarly $P(= 1 - P(A_{2m+1} > V_{2m+1}) > 1 - F(1) - \in$. (2.2.8)

Hence it follows from above that
$P(E_m U F_m) > 2(F(-1) - \in)(1 - F(-1) - \in) for \in > 0$.

Hence as $m \to \infty, P(E_m U F_m) \geq 2F(-1)\big(1 - F(-1)\big) > 0$.
Hence the proof.

Lemma 2.5

Out side a set of measure at most $\frac{C'}{k}$ the event $(E_m \cup F_m)$ occurs at least for C_k values of m. i.e out side a set of measure$\frac{C'}{k}$. $A_{2m} > V_{2m}\ and\ A_{2m+1} < -V_{2m+1}$.

Proof of the lemma

$(E_m \cup F_m)$ occurs with positive probability. (by Lemma 2.4)

Suppose $P(E_m \cup F_m) = \zeta_m > 0$

Then there exist an absolute constant$\eta > 0$, such that$P(E_m \cup F_m) = \zeta_m > \zeta$.

Let$\zeta = \sum_S \zeta_m$, where summation $\sum_S$ means that the summation is taken over all's pairs.

Applying Chebyshev's inequality, we have for $0 < \epsilon < \zeta_m$.

$P(|\zeta - E(\zeta)| \geq S\epsilon) \leq \frac{\mu_1}{k}$, there fore outside a set of measure $\frac{\mu_1}{k}$.

$|\zeta - E(\zeta)| < S\epsilon$, there fore, it follows that out side a set of measure$\frac{\mu_1}{k}$, $E_m \cup F_m$ occurs i.e either $A_{2m} > V_{2m}$ or $A_{2m+1} < -V_{2m+1}$

III. MAIN RESULTS

Proof of the theorem

Define $\xi_m = \begin{cases} 0 & if\ |R_{2m}| < V_{2m} and\ |R_{2m+1}| < V_{2m+1} \\ 1 & otherwise \end{cases}$

We have $f(x_{2m}) = A_{2m} + R_{2m}$ and $f(x_{2m+1}) = A_{2m+1} + R_{2m+1}$

$\zeta_m - \zeta_m\xi_m = 1$ only if $\zeta_m = 1\ and\ \xi_m = 0$ which implies the occurrence of one of the events.

$E\left(\sum_{m=m_0}^{k} \zeta_m\xi_m\right) \leq (k+1)\left\{C_1 e^{-M} + \left(\frac{C+1}{\alpha+1}\right)\frac{1}{\lambda^\alpha}\right\}$

Hence for $1 < \beta < 2$

$$P\left(\sum_{m=m_0}^{k} \zeta_m\xi_m > (k+1)\left(C_1 e^{-M} + \left(\frac{C+1}{\alpha+1}\right)\frac{1}{\lambda^\alpha}\right)\lambda^\beta\right) < \frac{C_4 k}{\lambda^{\alpha-\beta}} \qquad (3.3.1)$$

except for a set of measure $\leq \frac{1}{\lambda^\beta}$.

From (2.2.1), it follows that $M^{2k} \leq n \leq M^{2k+2}$

After some calculation

$\mu_1 \frac{logn}{logM} \leq k \leq \mu_2 \frac{logn}{logM}$, we have $M = \left[\frac{2^{\alpha+1} A e \lambda^\alpha}{B}\right] + 1$

It follows that

$$\mu_5 \frac{logn}{loglogn} \leq k \leq \mu_6 \frac{logn}{loglogn} \qquad (3.3.2)$$

Earlier we have to shown that

$\sum_{m=m_0}^{k} \zeta_m > \mu' k$ out side a set of measure $\frac{\mu'}{k}$.

Now $N_n > \sum_{m=m_0}^{k}(\zeta_m - \zeta_m\xi_m) > \mu k$, out side a set of measure at most.

but $\frac{\mu'}{k} + \frac{1}{\lambda^\beta} < \mu' \frac{logn}{loglogn} + \frac{1}{(logn)^\beta} \leq \mu' \frac{logn}{loglogn}$, since $(logn)^\beta > \frac{logn}{loglogn}$ as $(1 < \beta < 2)$

Hence $N_n > \mu \frac{logn}{loglogn}$

Similarly

except for a set of measure at most $\mu' \frac{loglogn}{logn}$.

REFERENCES

[1] Erdos, P and offord , A.C. On the number of real roots of a random algebraic equation, Proc. London. Math . Soc. (3) 18 (1956), 139-160.

[2] Kac, M. On the average number of real roots of a random algebraic equation, Bull. Amer. Math. Soc. 49 (1943) , 314 – 314 .

[3] Loeve, m. Probability Theory, Third edition (Van Nostrand Princeton ,1963) .

[4] Mishra, M.N. Nayak, N.N and pattanayak, S. Lower bound of the number of real roots of a random algebraic polynomial (I), Jour. Indian.Math.Soc. 45(1981), 1-25.

[5] Mishra, M.N. Nayak, N.N and pattanayak, S. Lower bound of the number of real root s of a random algebraic polynomial, Jour. Aust.Math.Soc. (Ser-A) 35, (1983), 18-27.

[6] Samal, G and Mishra, M .N. On the lower bound of the number of real roots of a random algebraic equation with infinite variance (I), Proc. Amer. Math. Soc. 33 (1972), 523-528.

[7] Samal, G and Mishra, M .N. On the lower bound of the number of real roots of a random algebraic equation with infinite variance (II), Proc. Amer. Math. Soc. 33 (1972), 557-563.

[8] Samal, G and Mishra, M .N. Real zeros of a random algebraic polynomial Quar. Jour. Math. Oxford, 2(1973), 169-175.

[9] Shimizu,.R On the domain of attraction of semi stable distributions, Ann. int stat.math22(1970)245-255

[10] Titchmarsh, E.C. The theory of Functions, 2nd end, The English Language Book Society, (Oxford University Press, 1939).

AUTHORS

First Author – Dr.N.N.Nayak , Ex-Prof in Mathematics, Orissa University of Agriculture & Technology, Bhubaneswar-3, India, Plot No.1242/2171, Siripur Nuasshi

Second Author – N.K.Sahoo, Faculty of Science and Technology, The ICFAI University Tripura, Kamalghat, Sadar, 799210, India