Exploring and Analyzing Students' Algebraic Misconceptions and Errors

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Abstract

This research based paper presents the exploring and analyzing students' algebraic misconceptions and errors (M/Es). Along with minute observation of students' classroom tasks and their peer groups' algebraic activities, the exploration of students' M/Es was carried out by conducting in-depth interview with students (based on their work sheets). The study had analyzed misconceptions and errors of the students of grade eighth. It was carried out especially in the area of algebraic content. This study has revealed altogether 63 different types of M/Es which have been framed up into seven categories. While diagnosing the origin, nature, patterns and causes of M/Es, some non-cognitive causes of M/Es have also been traced out. The study showed that diagnosis of students' M/Es on time should be the fundamental system of teaching/learning (T/L) school algebra.

Keywords: Exploring, analyzing, misconceptions, errors, causes, patterns, constructivism, fallibilism

I. INTRODUCTION

The mathematics education is a body of knowledge and area of science with its own terminology, symbolism, contents, theorems and T/L methodology. Students must know the mathematical language with lots of mathematical concepts, theories, and relations with its own syntax, grammar, words, phrases, sentences, symbols, graphs, visuals, models etc. But most of the students pass their time by listening and reading instead of writing, thinking, analyzing and using the mathematical language. As a result, they miss the logical power and cannot develop a creative power to think.

In this situation, students follow rote learning, and their mathematical knowledge is developed with a lot of misconceptions. For example, students start solving a problem but cannot reach the end with correct answer as they are lost somewhere in the middle of the process where they are misguided due to misconceptions and hence produce errors. When students study and explain incorrect skills, they directly meet misconceptions and are likely to maintain incorrect ways of thinking about problems (Warren & Cooper, 2006). If such misconceptions are not addressed as early as possible before they get matured, students integrate useless new information into their immature conceptions which cause further misconceptions and errors.

Piaget's (1970) fundamental assumption is that knowledge is a process, not a state. So, misconceptions are acquired during students' learning process. Actually, conceptions and misconceptions grow together, side by side. Students construct their misconceptions through their own experience.The intuitive interpretation based on a primitive, limited and unrelated individual

experience disturbs learning process and distorts correct conceptions which cause misconceptions (Fischbein, 1994). They occur while obtaining, processing, retaining and reproducing the information along with assimilating and accommodating new conceptions into existing conceptions. Since M/Es are emotionally attached with students, they are not easy to dislodge and remove (Egodawatte, 2011).

To err is human; however, the main point is concerned with what conclusions are drawn and learnt from errors. If errors are committed, it is said that they arise because the children are thinking and not because they are careless. The teacher should generate an environment of engaging students for the correction of errors with reasons instead of correcting them mechanically. Thus, teachers have to accept students' errors for the purpose of analysis.

The analysis of error pattern provides us an effective and efficient method for pinpointing the specific misconceptions and problems that students are having while solving problems. By investigating students' M/Es, the teacher can provide instruction targeted to their area of need. In this regard, Upadhyay (2017) claimed that if one could find out weaknesses and misconceptions of students, more than half of the problems of T/L tasks are done. Similarly, Ausubel (1968) said that if he had to reduce the entire body of knowledge in the field of educational psychology to just one principle, he would say: The most important single factor influencing learning is to ascertain what the learner already knows, and teach him/her accordingly. It shows the crucial role of M/Es in a whole learning process.

Students' errors are the symptoms of misconception like fever for the disease. So, it was imperative to identify students' conceptions, procedures, reasoning, and how students' wrong responses and errors expose their misconceptions. Further, it was more important to carry out such a research in algebraic area as it is a gate keeper for all branches of mathematics. However, a clear understanding is needed as regards how to identify, diagnose and address students' M/Es so that students would have a learning algebra meaningfully. For these reasons, I had conducted the study with two major research questions.

II. RESEARCH QUESITONS

The study was conducted to bestow the answers of the following two major questions.

(i) What are the students' algebraic misconceptions and errors?
(ii) What are the causes of students' algebraic misconceptions and errors?

III. THEORETICAL FRAMEWORK OF THE STUDY

One of the main things that makes human being different from other animals is the human's ability to learn very complex behaviors and skills. Though, the knowledge constructing routes are very complicated and often include backward and forward movements, suggesting that frequently making errors may be an unavoidable part of the learning process which does not mean that a learning strategy got unsuccessful (Campbell, 2009, cited in Kshetree, 2015). Further, he added that errors are logically consistent and rule-based rather than random. Thus, exploring errors presents an opportunity for uncovering the mental representations with mathematical reasoning and misconceptions. For this, the researcher considered different perspectives of constructivism such as

neural network, psychological, generate and degenerate, and buggy algorithm in learning algebra. They provided the various root causes of originating M/Es.

In addition, the different scholars and mathematicians have considered the nature of mathematics and its ontology, epistemology and axiology differently (Ernest, 1995). Platonist philosophers on the extreme right conclude that mathematics is objective, unhistorical, unchallengeable truth existing independent of consciousness. But the fallibilist philosophers argue that mathematics is subjective; socio-cultural, M/Es prone, and no mathematical truth can exist outside what learners have consciously constructed in their minds. Along with these different philosophical lenses, the researcher has used especially fallibilist view to study students' algebraic learning ways, committing M/Es including their remedy.

Actually, it is important to understand student's inclination towards different mathematical philosophies (though they may not clearly express them) to find out their M/Es particularly while interviewing them. Their views enable teachers to help students more as the teachers make world-view of mathematics from the student's point of view. Sometimes, there may be the need for modification in the learner's philosophies if they are found to be unhelpful.

While students are constructing concepts, they sometimes construct incomplete, immature, alternative and transitional concepts (Makonye, 2013, cited in Kshetree, 2015). Those concepts constructed by students may be fully correct or partially correct or completely wrong. In this way, the students' construction of algebraic concepts is a fallible process. Though students used intuition and trial and error process while guessing mathematical results and checking them, fallibilist approach was the main philosophy adopted in this research study with the assumption that the students construct algebraic concepts in their own endeavors.

In this way, out of five major mathematical philosophies, fallibilism was taken as a guiding philosophy for this research study. In order to study, analyze and diagnose the nature, origin, causes and different forms of M/Es, the theoretical foundations considered in this study included constructivism through different perspectives such as neural network, psychological, generate and degenerate, and buggy algorithm.

IV. CONCEPTUAL FRAMEWORK OF THE STUDY

As per the nature of the study, the researcher followed the qualitative research design. The qualitative information was collected through in-depth interview based on students' test and class-work copies, which was supported by the information gathered from classroom observations of T/L practices and field notes. After reviewing theoretical and empirical literatures, the exploration of students' M/Es was planned to carry out through the fallibilism and constructivism theories whereas they were analyzed through thematic approach by using critical judgment process. In this way, for the exploring and analyzing students' misconceptions and errors, the blue print of the researcher's mental map was sketched in the form of a 'conceptual framework' for this study.

V. METHODOLOGY

Research Design

As per the nature of information and data of the study, the research design adopted was qualitative. Creswell (2012) claims that interviewer gains holistic picture in qualitative research by analyzing words and phrases, reporting the views of the informants in detail, and carrying out research in a natural setting. Regarding collecting and processing method of data, the researcher collected and analyzed the data for the purposes of obtaining a deeper understanding of the phenomenon. The researcher was not limited in interview rather he was interacted with the students based on their class works and some tests in addition to tracing regular observations and making field notes. The notes were also made with reference to discussion held with mathematics teachers and students. Further, he combined these research strategies which were found to be mutually reinforcing.

Population and Sampling

As per the disposition of the study, a single visit and contact with students could not work well, so the researcher needed to have field work for a long time. Thus, the researcher had to be confined in few of the schools of Kathmandu Valley. So, the population of the study was all the community schools of the Valley.

At first, the researcher purposively made a list of seven schools then he visited and observed those schools. In the visits, he consulted the head teachers, teachers, and students for their consent, interest and cooperation. Then he selected four from those seven schools. Out of altogether 95 students, the study analyzed the test scores of only those 80 students who were regular and took part in the tests, interaction and interviews taken, time and again.

Selection of students' standard. Regarding selection of T/L class, the researcher had reviewed some relevant literatures. According to Booth (1988), the learning of algebra has received more attention at the lower secondary level school which is the critical stage of transition from arithmetic to algebra. Kieran (2007) claimed that most algebra studies have targeted learning of school algebra. Ginsburg (1997, cited in Kshetree, 2015) added that the results of the study of school algebra apply more to the students younger than high school age. So, it was important to know the M/Es that middle school students commit. Compared with the goal of arithmetic which has to find the answer, the focus of algebra was to find the general method and use algebraic symbols to express them in a general form. Further, Booth (1988) claimed that the structural representation, symmetric and transitive character means understanding equation as object which should become the goal of school algebra at the end of Basic Education level (eighth grade). That is why, the study was conducted among the students of eighth grade.

Development of Research Tools

In order to collect the required data for this study, the researcher prepared, standardized and intervened important research tools which include: Observation forms, field notes, test items, and interview guideline. Then, the researcher consulted with senior researchers, subject experts, trainers and teachers for feedback, suggestions and necessary modifications. Some of the tools were adapted from other researches with some modifications as per the need of the research.

Observation checklist for exploring M/Es

The researcher had to observe the practices of students' algebraic knowledge and understanding translated into problem solving skills even in the peer groups. For this, out of three types of observation (informal observations, structured observations and interactive observational assessment), the researcher had followed the modality of interactive observational assessment. While observing group work activities, six behaviors of mathematization were captured as suggested by Millory (1992, cited in Upadhyay, 2001) which include; explaining something to the group, taking some action over task, engaging in reflection, offering some arguments for verification, appraising some works critically and acting as a more able peer.

Thus, in order to explore students' misconceptions and errors, an observation checklist based on Newmann's Error Analysis Tool (1983), Perso's Diagnostic model (1991) and Blessing's testing (2004) was prepared and used it by the researcher while observing classroom T/L practices. While observing class works of the students, an attention was given for groups' discussion, exchanging the peer groups' work, and allowing the students to reform their peer groups. After the class is over, it used to have review meetings among teachers and researcher. A field note was also updated appropriately to collect all the relevant information explored while observing class works and discussion taken with the students and teachers.

Field notes

The field notes contain the ongoing data that were being collected. They consisted of the descriptions of what was being experienced and observed, the observer's feeling and reactions to what were observed, and field-generated insights and interpretations as claimed by Newman (1983). In the same manner, the researcher had observed and experienced the different types of representations such as strategic solutions to the given problems, the way of content of the algebra transmitted, and appropriateness of the use of T/L aids including the response of teacher when a student gives a wrong answer to a question. The field notes were taken lively on T/L classroom practices and activities where researcher traced his insight regarding making meaning upon what students observe, hear, communicate, and take part in peer works. In this way, the field notes strengthened and systematized the classroom observation tasks.

Development of test items

In order to identify and locate students' M/Es, the researcher developed and standardized the test tools. For this, he prepared the test items by using textbook, specification grid-chart, curriculum and teacher's guidebook developed and prescribed by the Curriculum Development Center (CDC). The specific misconceptions could easily be dealt with if algebra was seen through generalizing, forming and solving equations, and working with formulae. In this way, the test items were prepared based on Perso's

Diagnostic Test(1991) and Blessing's Algebraic Thinking Content Knowledge Test(2004). The test model was prepared on the basis of seven categories of students' M/Es. The test items included three categories of questions (knowledge for concepts, comprehension for process and application for academic skills) as per the Bloom's Taxonomy.

Interview guideline

In order to explore and fix the students' algebraic M/Es with reference to their test copies, an interview guideline was prepared and administered among the students of the sampled schools. The guideline was prepared being based on the conceptual understanding of the algebraic concepts, problem solving procedure and application of the knowledge in required steps of solving problems.

The interviews of the students were taken by the researcher himself. The interviews enabled the researcher to examine and obtain valid and reliable information about students' knowledge and understanding held in each test item of pre-test. More importantly, in order to explore students' M/Es, the interviews were used to follow up unexpected results or to validate other methods or to go deeper into the understanding of their reasons. For this purpose semi-structured open-ended questions were developed on the basis of the results of the test.

In order to know students' thinking patterns, conceptual status, constructions, interpretations and arguing ability, sequential questions were asked on the basis of their response in each stage. Further, unstructured interview with semi-structured questions was used as its purpose was to obtain a free flow of information, move forward and backward in the process of interviewing to clarify points, go over earlier points and again raise fresh questions, and create an atmosphere where the interviewee would feel free to come out with subjective and personal information. So, the interviews proceeded ahead more like a normal conversation, but with a purpose of locating students' M/Es.

Real interview

Before starting each interview, every student used to be informed about what was the objective of interview including expectation from the student. The initial conversation used to begin with, "I am going to ask you some of the questions based on your test answer-sheet. I would like you to try and explain to me your thinking while you are working. What I am interested in is how you are going to arrive at your answers. Sometimes I may ask you some short questions for further clarifications".

Further, the researcher made clear that it was quite acceptable to make any errors in the interview since the learners' thinking underlying their errors was more important than answers. Further, they were made to understand that the primary goal of this exercise was not to evaluate them or to offer them a mark. The interviews were taken on the basis of their answer sheets of test in order to identify their misconceptions in each typical and non-common answer. The interviews were also taken even in the correctly answered

questions in order to ensure their answer and confirm their existing knowledge. The duration of the interview was largely dependent on the responses given during the interview but did not last longer than thirty minutes.

In order to make them comfortable in delivering their opinion without any hesitation, a liberty of sitting two friends at a time for the interview was also permitted. As expected, the students were found more confident and comfortable in the interviews taken with pair of friends chosen by them. Actually, it was an idea developed in interview sites as seeing the weak students could not put their conceptual understanding clearly. In addition, two separate informal group discussions were also conducted among 7 students of one school and 9 students of another school. In the discussion, students were presented with their own test copies so that their ideas and arguments could be compared and exchanged among their friends. Further, few of the students were also interviewed informally while observing their group works.

In each interview, the researcher was interested in how students' incorrect responses could expose their M/Es. It helped further to explore their M/Es. In the end of each interview session, students' work sheets used in interview process were also collected. Further, each interview was audio taped and later transcribed and analyzed.

VI. ANALYSIS AND INTERPRETATION OF THE DATA

In order to identify students' common patterns of M/Es, it was taken in-depth interviews of the students based on their test and class work copies. Also at the time of observation, students' experiences were drawn by making them discuss and interact in peer groups, present each group's work in whole classroom, and notice teachers' guidance and support. The students were encouraged to put forth their reasons individually so that each one could explain how s/he arrived at the solution. Doing so was important than showing their answer sheets to the teacher in order to explore and verify students' M/Es identified from test-based interviews.

In order to organize the qualitative information systematically and standardize the supporting system through theorizing the findings of research, the thematic approach with critical judgment was applied. In this context, the study was focused on three areas of M/Es: Sources, causes and categories. The analysis of qualitative information obtained from the interviews, classroom observations of individual and peer groups' activities, and field notes was carried out with reference to theory of mathematical fallibilism, constructivism and findings of the previously carried out related researches. The M/Es were diagnosed based on the tools developed by Perso (1991), Blessing (2004) and Newmann (1983). Further, the data were categorically analyzed and interpreted as per their three levels of cognitive domain (conceptual, procedural, and application).

In this way, the students' M/Es were explored and triangulated through five different ways:

(i) Examination of test copies of all students,

(ii) In-depth interviews of the students based on their test answer sheets,

(iii) Scanned copies of class work and their analysis,

(iv) Field notes on observed peers' group tasks, interaction, and discussion,

(v) Reviewed researches and literatures

VII. FINDINGS OF THE STUDY

The findings regarding exploration and diagnosis of students' 63 different M/Es have been put forth in sequential order on the basis of their seven categories such as (i) building block of algebra, (ii) mis-manipulations, (iii) application, (iv) solving problems by patterns, (v) translation of word problems, (vi) generalization of algebraic concepts, and (vii) dealing symbolic expressions and equations. In addition, there were some miscellaneous proto type M/Es and their non-cognitive causes. The categorical findings have been specified, turn by turn, as follows.

Regarding the M/Es based on basic building block of algebra about the use of letters, symbols and variables, it was found that the students had few misconceptions such as: variables always have the same meanings and interpretations even in different contexts and situations, one variable is for one specific value only, variables are used to label name of words and objects, letters are meaningless symbols, and there is one-to-one correspondence of English alphabets with positive integers.

The M/Es related to mis-manipulation of algebraic symbols were mainly the perceiving incomplete answers as far as they see operating signs, taking equal sign as step maker, and missing some parts of terms and signs in the lack of comprehending procedural skills.

Incorrect application of rules and formula was another category of M/Es which consists of overgeneralization of rules, thinking backward in arithmetical way, and misuse of order of operations which forces to adopt inconsistent procedures.

With reference to use of numerical pattern and predictable visual to solve problems; the students were found misled by superficial structure of some algebraic formula and expressions, lacking discrimination ability of similar concepts, and having low confidence level in attempting given algebraic problems.

The common practices of M/Es with regard to solving word problems were related to understanding the problem, transformation of given language into algebraic equations, solving the problem correctly, and even encoding the solution. Further, they also had a problem in finding the relationship of variables to form equations; as a result, sometimes they guessed the answers and used trial and error method.

The overall image that emerged out from the findings of M/Es related to generalization of algebraic concepts was that students have lack of required concepts in formulation and use of general rules. Similarly, they had inter-subject messy concepts and even lack of the idea about where and how to start solving the problem.

Regarding the category of M/Es in solving problems using symbolic expressions, mainly three types of difficulties were found which caused M/Es. They were: lack of making the learning meaningful as a result following mechanical drill and practice, running through several steps to reach the so-called final step any way, and indulging into many mish-mashing concepts which deviated them from getting correct conceptions.

Furthermore, it was explored that students' M/Es were based on some psychological matters such as anxiety, hurriedness, poor attention, rigid thinking, recalling problem, nervousness, and time pressure including some technical parts like slips and carelessness. It was also observed in peer groups that some of the M/Es were based on students' own incomplete ideas, hazy instructions provided by teachers, some unclear sharing of ideas among friends, and use of own irrelevant life experience. However, students' M/Es were based on lack of conceptual clarity, low ability in comprehending procedural steps, and applying incorrect schemas in solving problems.

VIII. IMPLICATIONS OF THE STUDY

This study has drawn many prominent implications which can be used primarily by students and teachers in daily T/L practices of algebra. Further, the implications have been identified for subject experts and institutions as well.

For students, the study is much useful since the exploration of students' M/Es is based on their pre-existed knowledge and practices. Similarly, it identifies their learning schema and shortcomings, required treatments and determines its dose as per the nature of their M/Es which are essential ingredients to make the learning meaningful.

For teachers, they provide inputs as per the students' M/Es, ability, and interest where they encourage students to talk inside out and share with teachers. In this approach of T/L system, the role of teachers changes from talking to listening, describing to feed backing, relaying information to using creativity, being teacher centric to students centric, and as a whole lecturing system to facilitating the learning.

At expert level, it is expected to be equally applicable to policy makers of school education reform, curriculum developers, training package designers, examination experts, teacher educators, book writers, and trainers to name few of the representative professionals of education sector. This new approach of T/L through exploring and diagnosing students' M/Es can be a training material for pre-service and in-service mathematics teachers as well.

For institution, with regard to the intervention of this new approach in Nepalese classroom and context, the institutes will get overall conducive environment and tremendous improvements not only in mathematics but also in other subjects if they follow this approach of T/L system. However, it recommends some pre-requisites such as training for the teachers, specific lesson plans, and improved classroom situation. In this way, this new approach to T/L system did not only connect teachers and students for their betterment but also showed much significant benefits in the favor of institutions as a whole.

IX. CONCLUSION

The students and teachers together must reach the schemata of students to explore their M/Es. Teachers should not be anxious of students' M/Es because their wrong answers can guide reach to the origin of M/Es that they may be the best tools for crafting their learning experiences. The teachers' belief system in mathematics as absolute knowledge needs to be changed into

mathematical knowledge guided by fallibillism philosophy and human conscience. As a result, teachers could give importance to every reasoning of the students as the construction of knowledge needs own way, pace, experience, context and culture. Thus, exploring and analyzing students' misconceptions and errors should be the fundamental system of T/L algebra. Further, the study has also set up a norm in mathematics education like 'prevention of students' M/Es is better than their treatment'. Similarly, it has established a new knowledge as 'determine the ways and dose of treatment on the basis of students' diagnosed M/Es' in order to make T/L algebra meaningful.

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