Separation Axioms on (1,2)*-R*-Closed Sets in Bitopological Space

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Abstract- In the present paper, we introduce and the concept of (1,2)*-R*-T_i-space (for i = ½, 0, 1, 2) and we discuss some of their basic properties.

Index Terms- (1,2)*-R*-T1/2-space, (1,2)*-R*-T0-space, (1,2)*-R*-T1-space, (1,2)*-R*-T2-space.

Mathematical Subject Classification: 54D10

I. INTRODUCTION

An elaborate study on generalized closed sets were done by researchers after Levine [7] introduced this concept in 1970. The concept of regular continuous functions was introduced by Arya S. P. and Gupta R [1]. Regular generalized continuous functions was studied by Palaniappan N. and K.C. Rao [8]. C. Janaki and Renu Thomas [4] introduced (1,2)*- R*-closed sets in bitopological spaces. The purpose of this paper is to introduce and study (1,2)*-R*-separation axioms in bitopological spaces.

II. PRELIMINARIES

Throughout this paper, X and Y denote the bitopological spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) respectively on which no separation axioms are assumed.

Definition 2.1: A subset A of X is called τ_1,τ_2-open[4] if A = A_1∪A_2, where A_1 ∈ τ_1, A_2 ∈ τ_2. The complement of τ_1,τ_2-open set.

Definition 2.2: A subset A of a bitopological space X is
1) τ_1,τ_2-closure[5] of A denoted by τ_1,τ_2-cl(A) is defined as the intersection of all τ_1,τ_2-closed sets containing A.
2) τ_1,τ_2-interior[5] of A denoted by τ_1,τ_2-int(A) is defined as the union of all τ_1,τ_2-open sets contained in A.
3) (1,2)*-regular open[10] if A = τ_1,τ_2-int (τ_1,τ_2-cl(A)) and (1,2)*-regular closed[10] if A = τ_1,τ_2-cl(τ_1,τ_2-int(A)).

Definition 2.3: A subset A of a bitopological space (X, τ_1,τ_2) is called (1,2)*-regular semi open[9] if there is a (1,2)*-regular open set U such that U ⊆ A ⊆ (1,2)*-cl(U). The family of all (1,2)*-regular semi open sets of X is denoted by (1,2)*-RSO(X).

Definition 2.4: The union of all (1,2)*-regular open subsets of X contained in A is called(1,2)*-regular interior of A and is denoted by (1,2)*-rint(A) and the intersection of (1,2)*-regular closed subsets of X containing A is called(1,2)*-regular closure of A and is denoted by (1,2)*-rc(A).

Definition 2.5: A subset A of (X, τ_1, τ_2) is called (1,2)*-R*-closed[4] if rcl(A) ⊆ U whenever A ⊆ U and U is (1,2)*-regular semi open in(X, τ_1, τ_2). The complement of (1,2)*-R*-closed sets is (1,2)*-R*-open set [4]. The family of all (1,2)*-R*-closed subsets of X is denoted by (1,2)*-R*-C(X) and (1,2)*-R*-open subsets of X is denoted by (1,2)*-R*-O(X).

Definition 2.6: A function f : (X, τ_1,τ_2)→(Y,σ_1,σ_2) is called
1) (1,2)*-continuous[6] if f^1(V) is (1,2)*-closed in (X, τ_1, τ_2) for every (1,2)*-closed set V in (Y,σ_1,σ_2).
2) (1,2)*-R*-continuous[4] if f^1(V) is (1,2)*-R*-closed in (X, τ_1, τ_2) for every (1,2)*-R*-closed set V in (Y,σ_1,σ_2).
3) (1,2)*-R*-irresolute[4] if f^1(V) is (1,2)*-R*-closed in (X, τ_1, τ_2) for every (1,2)*-R*-closed set V in (Y,σ_1,σ_2).

Definition 2.7: A map f : (X, τ_1,τ_2)→(Y,σ_1,σ_2) is called
1) (1,2)*-closed map[6] if f(U) is (1,2)*-closed in Y for every (1,2)*-closed set U of X.
2) (1,2)*-R*-open map[3] if f(V) is (1,2)*-R*-open in Y for every (1,2)*-R*-open set V in X.
3) strongly(1,2)*-R*-open map[3] if f(V) is (1,2)*-R*-open in Y for every (1,2)*-R*-closed set V in X.

Definition 2.8: A space X is said to be
1) a (1,2)*-T_1-space if for any pair of distinct points x and y, there exists open sets G and H such that x ∈ G, y ∈ H and x ∉ H, y ∉ G.
2) a (1,2)*-T_2-space if for any pair of distinct points x and y, there exists disjoint open sets G and H such that x ∈ G and y ∈ H.

III. (1,2)*-R*-SEPARATION AXIOMS

In this section, we introduce and study separation axioms and obtain some of its properties.

Definition 3.1: A space X is called a (1,2)*-R*-T_1/2 space if every (1,2)*-R*-closed set is (1,2)*-regular closed.

Theorem 3.2: A bitopological space (X, τ_1,τ_2) is (1,2)*-R*-T_1/2 space if each singleton {x} of X is either (1,2)*-regular open or (1,2)*-regular closed.
Proof: Let $x \in X$. Assume $\{x\}$ is not $(1,2)^*$-regular closed, then clearly $(X-\{x\})$ is not $(1,2)^*$-regular open and $X-\{x\}$ is trivially $(1,2)^*$-R*-closed set.

Since $(X, \tau_1, \tau_2)$ is $(1,2)^*$-R*-T$_{1/2}$ space, every $(1,2)^*$-R*-closed set is $(1,2)^*$-regular closed.

And hence $(X-\{x\})$ is $(1,2)^*$-regular closed, and hence $\{x\}$ is $(1,2)^*$-regular open.

**Definition 3.3:** A space $X$ is said to be $(1,2)^*$-R*-T$_0$-space if for each pair of distinct points $x$ and $y$ of $X$, there exists a $(1,2)^*$-R*-open set containing one point but not the other.

**Theorem 3.4:** A bi-topological space $(X, \tau_1, \tau_2)$ is $(1,2)^*$-R*-T$_0$ space iff for each pair of distinct points $x, y$ of $X$, $(1,2)^*$-R*-cl$(\{x\}) \neq (1,2)^*$-R*-cl$(\{y\})$

Proof: Let $x$ and $y$ be distinct points of $X$. Since $X$ is a $(1,2)^*$-R*-T$_0$-space, there exists a $(1,2)^*$-R*-open set $G$ such that $x \notin G$ and $y \notin G$.

Consequently, $X-\{x\}$ is a $(1,2)^*$-R*-closed set containing $y$ but not $x$. But $(1,2)^*$-R*-cl$(\{x\})$ is the intersection of all $(1,2)^*$-R*-closed sets containing $y$. Hence $y \notin (1,2)^*$-R*-cl$(\{x\})$, but $x \in (1,2)^*$-R*-cl$(\{x\})$ as $x \in X-\{x\}$.

Therefore, $(1,2)^*$-R*-cl$(\{x\}) \neq (1,2)^*$-R*-cl$(\{y\})$.

Conversely, let $(1,2)^*$-R*-cl$(\{x\}) \neq (1,2)^*$-R*-cl$(\{y\})$ for $x \neq y$.

Then there exists at least one point $z \in X$ such that $z \notin (1,2)^*$-R*-cl$(\{x\})$.

We have to prove $x \notin (1,2)^*$-R*-cl$(\{y\})$, because if $x \notin (1,2)^*$-R*-cl$(\{y\})$, then $x \in (1,2)^*$-R*-cl$(\{y\})$.

Then $z \notin (1,2)^*$-R*-cl$(\{x\})$, which is a contradiction.

Hence $(1,2)^*$-R*-cl$(\{x\}) = \{x\}$, which is an $(1,2)^*$-R*-open set containing $x$ but not $y$. Hence $X$ is a $(1,2)^*$-R*-T$_0$-space.

**Theorem 3.5:** If $f: X \to Y$ is a bijection, strongly $(1,2)^*$-R*-open and $X$ is a $(1,2)^*$-R*-T$_0$-space, then $Y$ is also $(1,2)^*$-R*-T$_0$-space.

Proof: Let $y_1$ and $y_2$ be distinct points of $Y$. Since $f$ is bijective, there exists distinct points $x_1, x_2$ of $X$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since $X$ is a $(1,2)^*$-R*-T$_0$-space, there exists a $(1,2)^*$-R*-open set $G$ such that $x_1 \notin G$ and $x_2 \notin G$.

Therefore $y_1 = f(x_1) \notin f(G)$, $y_2 = f(x_2) \notin f(G)$.

Since $f$ is strongly $(1,2)^*$-R*-open function, $f(G)$ is $(1,2)^*$-R*-open in $Y$. Thus there exists a $(1,2)^*$-R*-open set $f(G)$ in $Y$ such that $y_1 \notin f(G)$ and $y_2 \notin f(G)$. Hence $Y$ is $(1,2)^*$-R*-T$_0$-space.

**Theorem 3.6:** If $f: X \to Y$ be an injective $(1,2)^*$-R*-irresolute function and $Y$ is $(1,2)^*$-R*-T$_0$-space, then $X$ is $(1,2)^*$-R*-T$_0$-space.

Proof: Let $x_1, x_2$ be two distinct points of $X$. Since $f$ is injective, there exists distinct points $y_1, y_2$ of $Y$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$.

Since $Y$ is $(1,2)^*$-R*-T$_0$-space, there exists $(1,2)^*$-R*-open set $U$ in $Y$ such that $y_1 \notin U$ or $y_2 \notin U$.

Therefore $x_1 \in f^{-1}(U)$, $x_2 \in f^{-1}(U)$.

Since $f$ is $(1,2)^*$-R*-irresolute, $f^{-1}(U)$ is $(1,2)^*$-R*-open set in $X$.
Theorem 3.11: If f: X→Y is (1,2)*-R*-irresolute injection function and Y is (1,2)*-R*-T1-space, then X is (1,2)*-R*-T2-space.

Proof: Let x1, x2 be two distinct points of X. Since f is injective, there exists distinct points y1, y2 of Y such that y1=f(x1) and y2=f(x2).

Since Y is (1,2)*-R*-T1-space, there exists disjoint (1,2)*-R*-open sets U and V in Y such that y1∈U, y2∈V and y1∉V, y2∉V.

ie. x1∈f⁻¹(U), x1∉f⁻¹(V) and x2∈f⁻¹(V), x2∉f⁻¹(U).

Since f is (1,2)*-R*-irresolute, f⁻¹(U), f⁻¹(V) are (1,2)*-R*-open sets in X.

Thus for two distinct points x1, x2 of X, there exists (1,2)*-R*-open sets f⁻¹(U) and f⁻¹(V) such that x1∈f⁻¹(U) and x2∈f⁻¹(V).

Therefore, X is (1,2)*-R*-T2-space.

Definition 3.12: A space X is said to be (1,2)*-R*-T1-space if for any pair of distinct points x and y, there exists disjoint (1,2)*-R*-open sets G and H such that x∈G and y∈H.

Theorem 3.13: If f: X→Y is (1,2)*-R*-continuous injection and Y is (1,2)*-T2-space, then X is (1,2)*-R*-T2-space.

Proof: Let f: X→Y be R*-continuous injection and Y is (1,2)*-T2-space. For any two distinct points x1, x2 of X, there exists distinct points y1, y2 of Y such that y1=f(x1) and y2=f(x2).

Since Y is (1,2)*-T2-space, there exists disjoint (1,2)*-open sets U and V in Y such that y1∈U, y2∈V and y1∉V, y2∉V.

ie. x1∈f⁻¹(U), x2∈f⁻¹(V).

Since f is (1,2)*-R*-continuous, f⁻¹(U), f⁻¹(V) are (1,2)*-R*-open sets in X.

Further, f is injective, f⁻¹(U)∩f⁻¹(V) = f⁻¹(U∩V) = f⁻¹(φ) = φ.

Thus for two distinct points x1, x2 of X, there exists disjoint (1,2)*-R*-open sets f⁻¹(U) and f⁻¹(V) such that x1∈f⁻¹(U) and x2∈f⁻¹(V).

Therefore, X is (1,2)*-R*-T2-space.

Theorem 3.14: If f: X→Y is (1,2)*-R*-irresolute injection function and Y is (1,2)*-R*-T2-space, then X is (1,2)*-R*-T2-space.

Proof: Let x1, x2 be two distinct points of X. Since f is injective, there exists distinct points y1, y2 of Y such that y1=f(x1) and y2=f(x2).

Since Y is (1,2)*-R*-T2-space, there exists disjoint (1,2)*-R*-open sets U and V in Y such that y1∈U and y2∈V.

ie. x1∈f⁻¹(U), x2∈f⁻¹(V).

Since f is (1,2)*-R*-irresolute injective, f⁻¹(U), f⁻¹(V) are disjoint (1,2)*-R*-open sets in X.

Thus for two distinct points x1, x2 of X, there exists disjoint (1,2)*-R*-open sets f⁻¹(U) and f⁻¹(V) such that x1∈f⁻¹(U) and x2∈f⁻¹(V).

Therefore, X is (1,2)*-R*-T2-space.

Theorem 3.15: In any topological space, the following are equivalent.

1) X is (1,2)*-R*-T2-space.

2) Let x∈X. For each x≠y there exists a (1,2)*-R*-open set U such that x∈U and y∉(1,2)*-R*-cl(U).

3) For each x∈X, {x}∩{(1,2)*-R*-cl(U)} is a (1,2)*-R*-open set in X and x∈U.

Proof: (1)⇒(2) Assume (1) holds.

Let x∈X and x≠y then there exists disjoint (1,2)*-R*-open sets U and V such that x∈U and y∈V.

Clearly, X−V is (1,2)*-R*-closed set. Since U∩V = φ, U⊂X−V.

Therefore, (1,2)*-R*-cl(U)⊂(1,2)*-R*-cl(X−V).

y∉X−V ⇔ y∉(1,2)*-R*-cl(X−V) and hence y∉(1,2)*-R*-cl(U) by the above argument.

(2)⇒(3) For each x≠y, there exists a (1,2)*-R*-open set U such that x∈U and y∉(1,2)*-R*-cl(U).

So y∉{x}∩{(1,2)*-R*-cl(U)}: U is a (1,2)*-R*-open set in X and x∈U={x}.

(3)⇒(1) Let x, y∈X and x≠y.

By hypothesis, there exists a (1,2)*-R*-open set U such that x∈U and y∉(1,2)*-R*-cl(U).

⇒there exists a (1,2)*-R*-closed set V such that y∉V.

Therefore y∈X−V and X−V is (1,2)*-R*-open set.

Thus there exists two disjoint (1,2)*-R*-open sets U and V−X such that x∈U and y∈X−V.

Therefore X is (1,2)*-R*-T2-space.

Theorem 3.16: Let (X, τ₁, τ₂) be a bitopological space, then the following statements are true:

1) Every (1,2)*-R*-T2-space is (1,2)*-R*-T1-space.

2) Every (1,2)*-R*-T1-space is (1,2)*-R*-T₀-space.

Proof: The proof is straightforward from the definitions.

REFERENCES
