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Separation Axioms on (1,2)*-R*-Closed Sets in Bitopological Space

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Abstract- In the present paper, we introduce and the concept of (1,2)*-R*-T_i-space (for i = ½, 0, 1, 2) and we discuss some of their basic properties.

Index Terms- (1,2)*-R*-$T_{1/2}$-space, (1,2)*-R*-T_0-space, (1,2)*-R*-T_1-space, (1,2)*-R*-T_2-space.

Mathematical Subject Classification: 54D10

I. Introduction

An elaborate study on generalized closed sets were done by researchers after Levine [7] introduced this concept in 1970. The concept of regular continuous functions was introduced by Arya S. P. and Gupta R [1]. Regular generalized continuous functions was studied by Palaniappan N. and K.C. Rao [8]. C. Janaki and Renu Thomas [4] introduced (1,2)*-R*-closed sets in bitopological spaces.

The purpose of this paper is to introduce and study (1,2)*-R*-separation axioms in bitopological spaces.

II. PRELIMINARIES

Throughout this paper, X and Y denote the bitopological spaces (X, τ_1, τ_2) and (Y,σ_1, σ_2) respectively on which no separation axioms are assumed.

Definition 2.1:A subset A of X is called $\tau_{1,2}$-open[1,4] if A = $A_i \cup B_i$, where $A_i \in \tau_1$, $B_i \in \tau_2$. The complement of $\tau_{1,2}$-open set.

Definition 2.2:A subset A of a bitopological space X is

1) $\tau_{1,2}$-closure[5] of A denoted by $\tau_{1,2}$-cl(A) is defined as the intersection of all $\tau_{1,2}$-closed sets containing A.
2) $\tau_{1,2}$-interior[5] of A denoted by $\tau_{1,2}$-int(A) is defined as the union of all $\tau_{1,2}$-open sets contained in A.
3) (1,2)*- regular open[10] if A= $\tau_{1,2}$-int ($\tau_{1,2}$-cl(A)) and (1,2)*-regular closed[10] if A= $\tau_{1,2}$-cl($\tau_{1,2}$-int(A)).

Definition 2.3: A subset A of a bitopological space (X, τ_1,τ_2) is called (1,2)*-regular semi open[9] if there is a (1,2)*-regular open set U such that $U \subset A \subset$ (1,2)*-cl(U). The family of all (1,2)*-regular semi open sets of X is denoted by (1,2)*-RSO(X).

Definition 2.4: The union of all (1,2)*-regular open subsets of Xcontained in A is called(1,2)*-regular interior of Aand is denoted by (1,2)*-rint(A) and the intersection of (1,2)*-regular closed subsets of X containing A is called(1,2)*-regular closure of A and is denoted by (1,2)*-rcl (A).

Definition 2.5:A subset A of (X, τ_1, τ_2) is called (1,2)*-R*-closed[4] if rcl(A) $\subset$ U whenever A $\subset$ U and U is (1,2)*-regular semi open in(X, τ_1, τ_2). The complement of (1,2)*-R*-closed sets is (1,2)*-R*-open set [4]. The family of all (1,2)*-R*-closed subsets of X is denoted by (1,2)*-R*-C(X) and (1,2)*-R*-open subsets of X is denoted by (1,2)*-R*-O(X).

Definition 2.6:A function f : (X, τ_1,τ_2)→(Y,σ_1,σ_2) is called

1) (1,2)*-continuous[6] if f^{-1}(V) is (1,2)*-closed in (X, τ_1, τ_2) for every (1,2)*-closed set V in (Y,σ_1,σ_2).
2) (1,2)*-R*-continuous[4] if f^{-1}(V) is (1,2)*-R*-closed in (X, τ_1, τ_2) for every (1,2)*-closed set V in (Y,σ_1,σ_2).
3) (1,2)*-R*-irresolute[4] if f^{-1}(V) is (1,2)*-R*-closed in (X, τ_1, τ_2) for every (1,2)*-R*-closed set V in (Y,σ_1,σ_2).

Definition 2.7: A map f : (X, τ_1,τ_2)→(Y,σ_1,σ_2) is called

1) (1,2)*-closed map[6] if f(U) is (1,2)*-closed in Y for every (1,2)*-closed set U of X.
2) (1,2)*-R*-open map[3] if f(V) is (1,2)*-R*-open in Y for every (1,2)*-open set V in X.
3) strongly(1,2)*-R*-open map[3] if f(V) is (1,2)*-R*-open in Y for every (1,2)*-R*-open set V in X.

Definition 2.8: A space X is said to be

1) a (1,2)*-T_1-space if for any pair of distinct points x and y, there exists open sets G and H such that $x \in G$, $y \notin G$ and $x \notin H$, $y \in H$.
2) a (1,2)*-T_2-space if for any pair of distinct points x and y, there exists disjoint open sets G and H such that $x \in G$ and $y \in H$.

III. (1,2)*-R* SEPARATION AXIOMS

In this section, we introduce and study separation axioms and obtain some of its properties.

Definition3.1:A space X is called a (1,2)*-R*-$T_{1/2}$ space if every (1,2)*-R*-closed set is (1,2)*-regular closed.

Theorem 3.2: A bitopological space (X, τ_1,τ_2) is (1,2)*-R*-$T_{1/2}$ space if each singleton {x} of X is either (1,2)*-regular open or (1,2)*-regular closed.

Proof: Let $x\in X$. Assume $\{x\}$ is not (1,2)*-regular closed, then clearly $(X-\{x\})$ is not (1,2)*-regular open and $X-\{x\}$ is trivially (1,2)*-R*-closed set.

Since (X, τ_1,τ_2) is (1,2)*-R*-$T_{1/2}$ space, every (1,2)*-R*-closed set is (1,2)*-regular closed.

And hence $(X-\{x\})$ is (1,2)*-regular closed, and hence $\{x\}$ is (1,2)*-regular open.

Definition 3.3: A space X is said to be (1,2)*-R*-T_0-space if for each pair of distinct points x and y of X, there exists a (1,2)*-R*-open set containing one point but not the other.

Theorem 3.4:A bitopological space (X, τ_1,τ_2) is (1,2)*-R*-T_0 space iff for each pair of distinct points x, y of X, (1,2)*-R*cl($\{x\}$) $\neq$ (1,2)*-R*cl($\{y\}$)

Proof: Let x and y be distinct points of X. Since X is a (1,2)*-R*-T_0-space, there exists a (1,2)*-R*-open set G such that $x\in G$ and $y\notin G$.

Consequently, $X-G$ is a (1,2)*-R*-closed set containing y but not x. But (1,2)*-R*-cl(y) is the intersection of all (1,2)*-R*-closed sets containing y. Hence $y\in$(1,2)*-R*-cl(y), but $x\notin$(1,2)*-R*-cl(y) as $x\notin X-G$.

Therefore, (1,2)*-R*-cl(x) $\neq$ (1,2)*-R*-cl(y).

Conversely, let (1,2)*-R-cl(x) $\neq$ (1,2)*-R*-cl(y) for $x\neq y$.

Then there exists atleast one point $z\in X$ such that $z\notin$(1,2)*-R*-cl(y).

We have to prove $x\notin$(1,2)*-R*-cl(y), because if $x\in$(1,2)*-R*-cl(y), then $\{x\}\subset$(1,2)*-R*-cl(y)$\Longrightarrow$ (1,2)*-R*-cl(x)$\subset$ (1,2)*-R*-cl(y)

Then $z\in$ (1,2)*-R*-cl(y), which is a contradiction.

Hence $x\notin$(1,2)*-R*-cl(y)$\Longrightarrow x\in X-$(1,2)*-R*-cl(y), which is an(1,2)*-R*-open set containing x but not y. Hence X is a (1,2)*-R*-T_0-space.

Theorem 3.5: If f: $X\rightarrow Y$is a bijection, strongly (1,2)*-R*-open and X is a (1,2)*-R*-T_0-space, then Y is also (1,2)*-R*-T_0-space.

Proof: Let y_1 and y_2 be distinct points of Y. Since f is bijective, there exists points x_1 and x_2 of X such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since X is a (1,2)*-R*-T_0-space, there exists a (1,2)*-R*-open set G such that $x_1\in G$ and $x_2\notin G$.

Therefore $y_1= f(x_1) \in f(G)$, $y_2= f(x_2) \notin f(G)$. Since f is strongly (1,2)*-R*-open function, f(G) is (1,2)*-R*-open in Y. Thus there exists a (1,2)*-R*-open set f(G) in Y such that $y_1\in f(G)$ and $y_2\notin f(G)$. Hence Y is (1,2)*-R*-T_0-space.

Theorem 3.6: If f: $X\rightarrow Y$be an injective(1,2)*-R*-irresolute function and Y is (1,2)*-R*-T_0-space, then X is (1,2)*-R*-T_0- space.

Proof: Let x_1, x_2 be two distinct points of X. Since f is injective, there exists distinct points y_1, y_2 of Y such that $y_1= f(x_1)$ and $y_2= f(x_2)$.

Since Y is (1,2)*-R*-T_0-space, there exists (1,2)*-R*-open set U in Y such that $y_1\in$ U or $y_2\notin$ U.

ie. $x_1\in f^{-1}(U)$ and $x_2\notin f^{-1}(U)$

Since f is (1,2)*-R*-irresolute, $f^{-1}(U)$ is(1,2)*-R*-open set in X.

Thus for two distinct points x_1, x_2 of X, there exists (1,2)*-R*-open set $f^{-1}(U)$ and such that $x_1\in f^{-1}(U)$, $x_2\notin f^{-1}(U)$.

Therefore, X is (1,2)*-R*-T_0-space.

Definition 3.7: A space X is said to be (1,2)*-R*-T_1-space if for any pair of distinct points x and y, there exists a (1,2)*-R*-open sets G and H such that $x\in G$, $y\notin G$ and $x\notin H$, $y\in H$.

Theorem 3.8: If arbitrary union of (1,2)*-R*-open space is (1,2)*-R*-open then, a space X is (1,2)*-R*-T_1-space iff singletons are (1,2)*-R*-closed sets.

Proof: Let X be (1,2)*-R*-T_1-space and $x\in X$. Let $y\in X-\{x\}$. Then for $x\neq y$, there exists (1,2)*-R*-open set U_y such that $y\in U_y$ and $x\notin U_y$.

Implies, $y\in U_y\subset X-\{x\}$.

That is $X-\{x\}= \cup\{U_y:y\in X-\{x\}\}$, which is a (1,2)*-R*-open set.

Hence $\{x\}$ is a (1,2)*-R*-closed set.

Conversely, suppose $\{x\}$ is (1,2)*-R*-closed set for every $x\in X$. Let $x,y\in X$ with $x\neq y$.

Now $x\neq y \Longrightarrow y\in X-\{x\}$. Hence $X-\{x\}$ is (1,2)*-R*-open set containing y but not x. Similarly, $X-\{y\}$ is (1,2)*-R*-open set containing x but not y. Therefore, X is a (1,2)*-R*-T_1-space.

Theorem 3.9: If f: $X\rightarrow Y$is strongly (1,2)*-R*-open bijective map and X is (1,2)*-R*-T_1-space, then Y is (1,2)*-R*-T_1- space.

Proof: Let f: $X\rightarrow Y$ be bijective and strongly (1,2)*-R*-open function. Let X be a (1,2)*-R*-T_1-space and y_1,y_2 be any two distinct points of Y.

Since f is bijective, there exists distinct points x_1, x_2 of X such that $y_1= f(x_1)$ and $y_2= f(x_2)$. Now, X being a (1,2)*-R*-T_1-space, there exists (1,2)*-R*-open sets G and H such that $x_1\in G$, $x_2\notin G$ and $x_1\notin H$, $x_2\in H$. Since $y_1= f(x_1) \in f(G)$ but $y_2= f(x_2) \notin f(G)$ and $y_2= f(x_2) \in f(H)$, $y_1= f(x_2) \notin f(H)$.

Now, f being strongly (1,2)*-R*-open, f(G) and f(H) are (1,2)*-R*-open subsets of Y such that $y_1\in f(G)$ but $y_2\notin f(G)$ and $y_2\in f(H)$, $y_1\notin f(H)$. Hence Y is (1,2)*-R*-T_1-space.

Theorem 3.10: If f: $X\rightarrow Y$is (1,2)*-R*-continuous injection and Y is (1,2)*-T_1-space, then X is (1,2)*-R*-T_1-space.

Proof: Let f: $X\rightarrow Y$ be R*-continuous injection and Y is (1,2)*-T_1. For any two distinct points x_1, x_2 of X, there exists distinct points y_1, y_2 of Y such that $y_1= f(x_1)$ and $y_2= f(x_2)$.

Since Y is (1,2)*-T_1-space, there exists open sets U and V in Y such that $y_1\in$ U, $y_2\notin$ U and $y_1\notin$ V, $y_2\in$ V.

ie. $x_1\in f^{-1}(U)$, $x_1\notin f^{-1}(V)$ and $x_2\in f^{-1}(V)$, $x_2\notin f^{-1}(U)$

Since f is (1,2)*-R*-continuous, $f^{-1}(U)$, $f^{-1}(V)$ are (1,2)*-R*-open sets in X.

Thus for two distinct points x_1, x_2 of X, there exists (1,2)*-R*-open sets $f^{-1}(U)$ and $f^{-1}(V)$ such that $x_1\in f^{-1}(U)$, $x_1\notin f^{-1}(V)$ and $x_2\in f^{-1}(V)$, $x_2\notin f^{-1}(U)$.

Therefore, X is (1,2)*-R*-T_1-space.

Theorem3.11: If f: X→Yis (1,2)*-R*-irresolute injection function and Y is(1,2)*-R*-T_1-space, then X is (1,2)*-R*-T_1-space.

Proof: Let x_1, x_2 be two distinct points of X. Since f is injective, there exists distinct points y_1, y_2 of Y such that $y_1= f(x_1)$ and $y_2= f(x_2)$.

Since Y is (1,2)*-R*-T_1-space, there exists (1,2)*-R*-open sets U and V in Y such that $y_1\in U$, $y_2\notin U$ and $y_1\notin V$, $y_2\in V$.
ie. $x_1\in f^{-1}(U)$, $x_1\notin f^{-1}(V)$ and $x_2\in f^{-1}(V)$, $x_2\notin f^{-1}(U)$
Since f is(1,2)*-R*-irresolute, $f^{-1}(U)$, $f^{-1}(V)$ are (1,2)*-R*-open sets in X.

Thus for two distinct points x_1, x_2 of X, there exists (1,2)*-R*-open sets $f^{-1}(U)$ and $f^{-1}(V)$ such that $x_1\in f^{-1}(U)$, $x_1\notin f^{-1}(V)$ and $x_2\in f^{-1}(V)$, $x_2\notin f^{-1}(U)$.
Therefore, X is (1,2)*-R*-T_1-space.

Definition 3.12:A space X is said to be (1,2)*-R*-T_2-space if for any pair of distinct points x and y, there exists disjoint (1,2)*-R*-open sets G and H such that x∈G and y∈H.

Theorem 3.13: If f: X→Yis (1,2)*-R*-continuous injection and Y is (1,2)*-T_2-space, then X is (1,2)*-R*-T_2-space.

Proof: Let f: X→Y be R*-continuous injection and Y is (1,2)*-T_2-space. For any two distinct points x_1, x_2 of X, there exists distinct points y_1, y_2 of Y such that $y_1= f(x_1)$ and $y_2= f(x_2)$.
Since Y is (1,2)*-T_2-space, there exists disjoint(1,2)*-open sets U and V in Y such that $y_1\in U$ and $y_2\in V$.
ie. $x_1\in f^{-1}(U)$, $x_2\in f^{-1}(V)$.
Since f is (1,2)*-R*-continuous, $f^{-1}(U)$, $f^{-1}(V)$ are (1,2)*-R*-open sets in X.

Further, f is injective, $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\varphi) = \varphi$.
Thus for two distinct points x_1, x_2 of X, there exists disjoint (1,2)*-R*-open sets $f^{-1}(U)$ and $f^{-1}(V)$ such that $x_1\in f^{-1}(U)$ and $x_2\in f^{-1}(V)$.
Therefore, X is (1,2)*-R*-T_1-space.

Theorem 3.14: If f: X→Yis (1,2)*-R*-irresolute injective function and Y is (1,2)*-R*-T_2-space, then X is (1,2)*-R*-T_2-space.

Proof: Let x_1, x_2 be two distinct points of X. Since f is injective, there exists distinct points y_1, y_2 of Y such that $y_1= f(x_1)$ and $y_2= f(x_2)$.

Since Y is (1,2)*-R*-T_2-space, there exists disjoint (1,2)*-R*-open sets U and V in Y such that $y_1\in U$and $y_2\in V$.
ie. $x_1\in f^{-1}(U)$, $x_2\in f^{-1}(V)$.
Since f is (1,2)*-R*-irresolute injective, $f^{-1}(U)$, $f^{-1}(V)$ are disjoint(1,2)*-R*-open sets in X.

Thus for two distinct points x_1, x_2 of X, there exists disjoint (1,2)*-R*-open sets $f^{-1}(U)$ and $f^{-1}(V)$ such that $x_1\in f^{-1}(U)$ and $x_2\in f^{-1}(V)$.
Therefore, X is (1,2)*-R*-T_2-space.

Theorem3.15: In any topological space, the following are equivalent.

1) X is (1,2)*-R*-T_2-space.
2) Let x∈X. For each x ≠ y there exists a (1,2)*-R*-open set U such that x∈U and y∉ (1,2)*-R*-cl(U).
3) For each x∈U,{x}=∩{(1,2)*-R*-cl(U):U is a (1,2)*-R*-open set in X and x∈U}

Proof: (1)⟹(2) Assume (1) holds

Let x∈X and x ≠ y then there exists disjoint (1,2)*-R*-open sets U and V such that x∈U and y∈V.
Clearly, X−V is (1,2)*-R*-closed set. Since $U \cap V = \varphi$, U⊂ X−V.
Therefore, (1,2)*-R*-cl(U) ⊂(1,2)*-R*-cl(X−V)
y∉ X−V ⟹ y ∉(1,2)*-R*-cl(X−V) and hence y ∉(1,2)*-R*-cl(U) by the above argument.
(2)⟹ (3) For each x ≠ y, there exists a (1,2)*-R*-open set U such that x∈U, y ∉(1,2)*-R*-cl(U)
So y ∉∩{(1,2)*-R*-cl(U):U is a (1,2)*-R* open set in X and x∈U}={x}
(3)⟹(1) Let x, y ∈X and x ≠ y
By hypothesis, there exists a (1,2)*-R*-open set U such that x∈U and y ∉(1,2)*-R*-cl(U).
⟹there exists a (1,2)*-R*-closed set V such that y ∉ V
Therefore y ∈ X−V and X−V is (1,2)*-R*-open set.
Thus there exists two disjoint (1,2)*-R*-open sets U and X−V such that x∈U and y∈X−V
Therefore X is (1,2)*-R*-T_2-space.

Theorem 3.16: Let (X, τ_1,τ_2) be a bitopological space, then the following statements are true

1) Every (1,2)*-R*-T_2-space is (1,2)*-R*-T_1-space.
2) Every (1,2)*-R*-T_1-space is (1,2)*-R*-T_0-space.

Proof: The proof is straight forward from the definitions

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