Fundamental Metric Tensors Fields on Riemannian Geometry with Applications to Integration of Tensor fields

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Abstract- In this paper is to introduce the idea of integration of tensor field .The definitions of indefinite and definite for tensor fields are similar to the analogous definitions for integrals functions differential calculus , and the definite integral tensor fields is also a tensor field of the sum type.

Index Terms- [1] The dual space of a vector space is defined as follows : the dual space V^* ,[2] $k \in N$ given a vector space $V_1,.....,V_k$ one can define a vector space $V_1 \otimes ..... \otimes V_k$ called their tensor product . [3]The definite integral tensor of f_t' on a closed interval $[a,b]$ is defined by the Newton –Leibniz formula $\int_a^b f_t' \, dt = f_t \,|_a^b = f(b) - f(a)$.[4] the indefinite integral of a vector field $[X,Y]$ is of the form $\int [X,Y]_t dt = Y_t + Y_0$, where Y is a differentiable vector field.

I. Introduction

The metric tensor is called positive-definite it assigns a positive vault every non-zero vector , a manifold equipped with a positive define metric tensor is a known a Riemannian manifold , having define vectors and one-form we can define tensor , a tensor of rank (m,n) also called (m,n) tensor , is defined to be scalar function of m one-forms and ν vectors that is linear in all of its argument, if follow at once that scalars tensors of rank (0,0) , for example metric tensor scalar product equation $\tilde{P}(\vec{V}) = \langle \tilde{P}, \vec{V} \rangle$ requires a vector and one-form is possible to obtain a scalar from vectors or two one-forms vectors tensor the definition of tensors , any tensor of $(0,2)$ will give a scalar form two vectors and any tensor of rank $(0,2)$ combines two one-forms to given $(0,2)$ tensor field g_x called tensor the g_x^{-1} inverse metric tensor , the metric tensor is a symmetric bilinear scalar function of two vectors that g_x and g_x is returns a scalar called the dot product .

(1.1) $$g(\vec{V},\vec{W}) = \vec{V}.\vec{W} = \vec{W}.\vec{V} = g(\vec{W},\vec{V}) .$$

Next we introduce one-form is defined as linear scalar function of vector $\tilde{P}(\vec{V})$ is also scalar product $\tilde{P}(\vec{V}) = \langle \tilde{P}, \vec{V} \rangle$ one-form $\tilde{p}$ satisfies the following relation.

(1.2) $$\tilde{P}(a\vec{V} + b\vec{W}) = \langle P, a\vec{V} + b\vec{W} \rangle = a\langle \tilde{P}, \vec{V} \rangle + b\langle \tilde{P}, \vec{W} \rangle = a\tilde{P}(\vec{V}) + b\tilde{P}(\vec{W})$$

and given any two scalars a and b and one-forms $\tilde{P}, \tilde{Q}$ we define the one-form $a\tilde{P} + b\tilde{Q}$ by.

(1.3) $$(a\tilde{P} + b\tilde{Q})(\vec{V}) = \langle a\tilde{P} + b\tilde{Q}, \vec{V} \rangle = a\langle \tilde{P}, \vec{V} \rangle + b\langle \tilde{Q}, \vec{V} \rangle = a\tilde{P}(\vec{V}) + b\tilde{Q}(\vec{V})$$

and scalar function one-form we may write $\langle \tilde{P}, \vec{V} \rangle = \tilde{P}(\vec{V}) = \vec{V}(\tilde{P})$, for example $m = 2$, $n = 0$ and $T(a\tilde{P} + b\tilde{Q}, c\tilde{R} + d\tilde{S}) = acT(\tilde{P},\tilde{R}) + adT(\tilde{P},\tilde{S}) + bcT(\tilde{Q},\tilde{R}) + bdT(\tilde{Q},\tilde{S})$ tensor of a given rank form a liner algebra mining that a liner combinations of tensor rank (m,n) is also a tensor rank (m,n) , and tensor product of two vectors A and B given a rank $(2,0)$, $T = \vec{A} \otimes \vec{B}$, $T(\tilde{P},\tilde{Q}) \equiv \vec{A}(\tilde{P}).\vec{B}(\tilde{Q})$ and $\otimes$ to denote the tensor product and non commutative $\vec{A} \otimes \vec{B} \neq \vec{B} \otimes \vec{A}$ and $\vec{B} = c\vec{A}$ for some scalar , we use the symbol $\otimes$ to denote the tensor product of any two tensor e.g $P \otimes T = \tilde{P} \otimes \tilde{A} \otimes \tilde{B}$ is tensor of rank $(2,1)$. The tensor fields in inroad allows one to the tensor algebra $A_R(T_pM)$ the tensor spaces obtained by tensor protects of space R , T_pM and $T^*{}_pM$ using tensor defined on each point $p \in M$ field for example M be n-dimensional manifolds a differentiable tensor $t_p \in A_R(T_pM)$ are same have differentiable components with respect , given by tensor products of bases $\left(\frac{\partial}{\partial x^k}\right)_p \subset T_p M \ , k = 1,....,n$ and $\left(dx^k\right)_p \subset T^*{}_p M$ induced by all systems on M . Laplace-"Bltrami operator" plays a fundamental role in Riemannian geometric in real applications smooth metric surface is usually as triangulated mesh the manifold including mesh parameterization segmentation let V is the space whose elements are linear functions from V^* is denote its dual space , we denote the of $\sigma(V^*)$ then $\sigma : V \to R$ for the any $v \in V$ we denote the value of σ on v by $\sigma(V)$ or by $\langle v, \sigma \rangle$.

Tensor Riemannian Geometry . A C^∞ covariant tensor field of order r on C^∞ manifold M is function M is assigns to each $p \in M$ an element φ_p of $f^r(T_p M)$ and which has additional property that given any $(X_1, X_2,....., X_r)$,C^∞ vector fields on an open subset U of M , then $\varphi_p(X_1,...., X_r)$ is C^∞ function on U , defined by $(X_1, X_2,....., X_r), P = \varphi_p(X_{1p},...., X_{rp})$ we denote by $f^r(M)$ the set of all C^∞ covariant tensor fields of order r on M . For each $\varphi \in f^r(U)$ including the restriction to U of any covariant tensor field on M , has a unique expression form.

(1.4) $$\varphi = \sum_{i1},...,\sum_{ir} a_{i1},...,a_{ir}\left(w^{ir} \otimes,...., w^{ir}\right)$$

where at each point $U, a_{i1},..,_{ir}$ are $\varphi(E_{i1},..., E_{ir})$ are the component of φ in the basis $\left(w^{ir} \otimes,.....,\otimes w^{ir}\right)$ and is C^∞ function on U . The tangent space $T_p M$ is defined as the vector space spanned by the tangents at p to all curves passing through point p in the manifold M , and the cotangent $T_p^* M$ of a manifold at $p \in M$ is defined as the dual vector space to the tangent space $T_p M$, we take the basis vectors $E_i = \left(\frac{\partial}{\partial x^i}\right)$ for $T_p M$ and we write the basis vectors $T_p^* M$ as the differential line elements $e^i = dx^i$ thus the inner product is given by $\langle \partial / \partial x, dx^i \rangle = \delta_i^j$.

A alternating covariant tensor field of order r on M will be called an exterior differential form of degree r , or some time simply r-form , the set $\Lambda^r(M)$ of all such forms is a subspace of $f^r(M)$, for example $\Lambda(M)$ denote the vector space over R of all exterior differential forms , then for $\varphi \in \Lambda^r(M)$ and $\psi \in \Lambda^s(M)$ the formula $(\varphi \wedge \psi)_p = \varphi_p \wedge \psi_p$ defines an associative product satisfying $(\varphi \wedge \psi) = (-1)^{rs}(\psi \wedge \varphi)$ with this product $\Lambda(M)$ is algebra over R if $f \in C^\infty(M)$ we also have

(1.5) $$f(\varphi\psi) = \varphi \wedge (f\psi) = \psi \wedge (f\varphi)$$

is a field of co frames on M or an open set U of M , an oriented vector space is a vector space plus an equivalence class of allowable bases choose a basis to determine the orientations those equivalents to will be called oriented or positively oriented bases or frames this concept is related to the choice of a basis Ω of $\Lambda^n(V)$, say that M is oriented if is possible to define a C^∞ n-form Ω on Ω which is not zero at any point in which case M is said to be oriented by the choice .

II. TENSOR ON A VECTORS SPACE

2.1 Tensors

In this section some fundamental constructions for a real vector space V are introduced . The dual space V^* the tensor spaces $T^k(V)$ and the alternating tensor spaces $A^k(V)$. The presentation is based purely on linear algebra , and it is independent of all the following , where we shall apply the theory of the present section to the study of manifolds. The linear space V will then be the tangent space $T_p M$ at a given point. Let V be a vector space over R .For our purposes only finite dimensional space are needed , so we shall assume dim. $V \leq \infty$ whenever it is convenient .

2.2 The dual space

We recall that the dual space of a vector space is defined as follows : the dual space V^* is the space of linear maps $\zeta : V \to R$ a liner map $\zeta \in V^*$ is often called a linear form equipped with the V^* becomes a vector space on its own. The basic theorem about V^* ,for V finite dimensional , is the following .

Definition 2.2.1

The dual space V^* is the space of linear maps $\zeta : V \to R$ a linear map $\zeta \in V^*$ is often called a linear form . Equipped with the natural algebraic operations of addition and scalar multiplication of maps V^* becomes a vector space on its own. The basic theorem about V^* for V finite dimensional is the following .

Theorem 2.2.2

Assume $\dim V = n \in N$ and let $e_1,....,e_n \in R$ be a basis (i) For each $i = 1,....,n$ an element $\zeta_i \in V^*$ is defined by $\zeta_i(a_1e_1 + ... + a_ne_n) = a_i$, $a_1,....,a_n \in R$. (ii) the elements $\zeta_1,....,\zeta_n$ form a basis for V^* (called the dual basis)

Proof :

Is easy for (ii) notice first that tow linear forms on vector space are equal, if they agree on each element of a basis . Notice also that it follows from the definition ζ_i that $\zeta_i(e_j) = \delta_{i,j}$ let $\zeta \in V^*$, then

$$\zeta = \sum_{i=1}^{n} \zeta(e_i)\zeta_i$$

Because the two sides agree on each e_j .This shows that the vectors $\zeta_1,......,\zeta_n$ space V^* . They are also linearly independent, for $\sum_i b\zeta = 0$ then $b_j = \sum_i b_i\zeta_i(e_j) = 0$ for all j .

Corollary 2.2.3

If dim $V = n$ then dim V^* if a linear form $\zeta \in V^*$ satisfies that $\zeta(V) = 0$ for all $v \in V$, then by definition $\zeta = 0$ then similar for elements $v \in V$ needs.

Corollary 2.2.4

Let $v \in V$ if $v \neq 0$ then $\zeta(v) \neq 0$ for some $\zeta \in V^*$.

Proof :

For simplicity we assume dim $v \prec \infty$ (although the result is true in general) . Assume $v \neq 0$. Then there exists a basis $e_1,....,e_n$ for V . This can be seen from the following theorem , which shows the elements of V^* can be used to detect whether a given belongs to a given subspace .

2.3 The dual of a linear map

Let V, W be vector space over R , and let $T : V \to W$ be liner . By definition, the dual map $T^* : W^* \to V^*$ takes a linear form $\eta \in W^*$ to its pull-back by T , that is $T^*(\eta) = \eta \circ T$. It is easily seen that T^* liner. For example assume V and W are finite dimensional and let $T : V \to W$ be linear . If T is represented by a matrix (a_{ij}) with respect to the dual base.

Lemma 2.3.1

Assume V and W are finite dimensional, and let $T : V \to W$ be linear . If T is represented by a matrix $(e_{j,i})$ with respect to some given bases then T^* is represented by the transposed matrix $(e_{j,i})$ with respect to the dual spaces .

Proof :

Let $e_1,....,e_n$ and $f_1,...., f_m$ denote the given base for V and W the fact that T is represented by $(a_{i,j})$ is expressed in the equality $Te_j = \sum_i a_{i,j} f_i$. Let $\zeta_1,....,\zeta_n$ and $\eta_1,....,\eta_m$ denote the dual bases for V^* and W^* then $a_{i,j} = \eta_i(Te_j)$ we now obtain with $\zeta = T^*{}_{\eta k}$ that .

$$T^*{}_{\eta k} = \sum_j T^*{}_{\eta k}(e_j)\zeta_j = \sum_j \eta k(Te_j)\zeta_j = \sum_j a_{k,j}\zeta_j$$

2.4 Tensors vector spaces

We now proceed to define tensors . Let $k \in N$ given a vector space $V_1,.....,V_k$ one can define a vector space $V_1 \otimes ..... \otimes V_k$ called their tensor product . The element of this vector space are called tensors with the situation where the vector space $V_1,.....,V_k$ are all equal to the same space. In fact the tensor space T^kV we define below corresponds to $V^*{}_1 \otimes ..... \otimes V^*{}_k$ in the general notation. And we define $V^k = V \times .... \times V$ be the Cartesian product of k copies of V .A map φ from V^k to a vector space U is called multiline if in each variable separately (i.e with the other variables held fixed) .

Definition 2.4.1

Let $V^K = V \times ..... \times V$ be the Cartesian product of k copies of V . A map φ from V^k to a vector space U is called multiline if it is linear in each variable separately (i.e with the other variables held fixed)

Definition 2.4.2

A(covariant) k-tensor on V is a multilinear map $T : V^k \to R$. The set of k-tensors on V is denoted $T^k(V)$. In particular, a 1-tensor is a linear form , $T^1(V) = V^*$. It is convenient to add the convention that $T^0(V) = R$. The set $T^k(V)$ is called tensor space, it is a vector space because sums and scalar products of multilinear maps are again multilinear.

2.5Alternating tensors

Let V be a real vector space . In the preceding section the tensor spaces T^kV were defined , together with the tensor product $(S,T) \to S \otimes T$, $T^k(V) \times T^l(V) \to T^{k+l}(V)$ there is an important construction of vector spaces which resemble tensor powers of V , but for which there is a more refined structure, These are the so-called exterior powers V , which play an important role in differential geometry because the theory of differential forms is built on them. They are also of importance in algebraic topology and many other fields. A multiline map $\varphi : V^k = V \times .... \times V \to U$ where $k \geq 1$ is said to be alternating if for all $v_1,......,v_k$ are inter-changed that is $\varphi(v_1,......,v_i,....,v_k) = -\varphi(v_i,.....,v_j,.....,v_i,....,v_k)$ since every permutation of numbers $1,......,k$ can be decomposed into transpositions, it follows that $\varphi(v_{\sigma 1},....,v_{\sigma k}) = \operatorname{sgn} \varphi(v_1,.....,v_k)$ for all permutations $\sigma \in S_k$ of the numbers $1,.....,k$ .For example let $V = R^3$ the vector product $(v_1,v_1) \to v_1 \times v_2 \in V$ is alternating for $V \times V \to V$ .And let $V = R$ the $n \times n$ determinant is multinear and alternating in its columns, hence it can be viewed as an alternating map $(R^n)^n \to R$.

Lemma 2.5.1

Let $\varphi : V^k \to U$ be multilinear . The following conditions are equivalent : (a) φ is alternating . (b) $\varphi(v_1,...,v_k) = 0$ whenever two of vectors $v_1,....,v_k$ are linearly dependent.

Proof :

(a) $\Rightarrow$ (b) if so implies $\varphi(v_1,....,v_k) = 0$,(a) $\Rightarrow$ (b) consider for example the interchange of v_1 and v_2 . By linearity $0 = \phi(v_1 + v_2, v_1 + v_2, ...) = \varphi(v_1,v_2,....)\varphi(v_1,v_2,.....) + \varphi(v_1,v_2,.....) + \varphi(v_1,v_2,.....) = \varphi(v_1,v_2,....) + \varphi(v_1,v_2,.....)$ If follows that $\varphi(v_1,....,v_k) = -\varphi(v_1,v_2,....)$ (a) $\Rightarrow$ (c) if the vector $(v_1,...,v_k)$ are linearly dependent then one of then can be written as a linear combination of the others. It then follows that $\varphi(v_1,....,v_k)$ is a linear combination of terms in each of which some v_i appears twice (a) $\Rightarrow$ (b) obvious . In particular , if $k \geq \dim .V$ then every set of k vectors is linearly dependent, and hence $\varphi = 0$ is the only alternating map $V^k \to U$.

Definition 2.5.2

An alternating k-form is an alternating k-tensor $V^k \to R$ the space of these is denoted $A^k(V)$, it is a linear subspace of $T^k(V)$

Theorem 2.5.3

Assume dim $V = n$ with $e_1,....,e_n$ a basis . let $\zeta_1,....,\zeta_n \in V^*$ denote the dual basis . The elements $\zeta_1 \otimes .... \otimes \zeta_{i,k}$ where $I = (i_1,....,i_k)$ is an arbitrary sequence of k numbers in $\{1,....,n\}$,form a basis for $T^k(V)$.

Proof :

Let $T_1 = \zeta_{i1} \otimes ...... \otimes \zeta_{ik}$. Notice that if $J = (j_1,....., j_k)$ is another sequence of k integers , and we denote by e_j the element $e_{j1},....,e_{jk} \in V^k$ then $T_I(e_j) = \delta_{Ij}$ that is $T_I(e_j) = 1$ if $J = I$ and 0 otherwise . If follows that the T_I are linearly independent, for if a liner combination $T = \sum_I a_I T_I$ is zero, then $a_j = T(e_j) = 0$. It follows from the multilinearity that a k-tensor is uniquely determined by its values on all elements in V^k of the form e_j . For any given k-tensor T we have that the k-tensor $\sum_I T(e_I)T_I$ agrees with T on all e_j hence $\sum_I T(e_I)T_I$ and we conclude that the T_I span $T^K(V)$.

2.6 The wedge product

In analogy with the tensor product $(S,T) \to S \otimes T$ form $T^k(V) \times T^l(V) \to T^{k+1}(V)$, there is a construction of a product $A^k(V) \times A^l(V) \to A^{k+1}$ since tensor products of alternating tensors are not alternating , it does not suffice just to take $S \otimes T$.

Definition 2.6.1

Let $S \in A^k(V)$ and $T \in A^l(V)$. The wedge product $S \wedge T \in A^{k+1}(V)$ is defined by $S \wedge T = ALt(S \otimes T)$.Notice that in the case $k = 0$,where $A^k(V) = R$, the wedge product is just scalar multiplication .

Example 2.6.2

Let $\eta_1, \eta_2 \in A^1(V) = V^*$ then by definition $\eta_1 \wedge \eta_2 = 1/2(\eta_1 \otimes \eta_2 - \eta_2 \otimes \eta_1)$ since the operator . Alt is linear the wedge product depends linearly on the factors S and T . It is more cumbersome to verify the associative rule for $\wedge$. In order to do this we need the following .

Lemma 2.6.3

Let $R \in A^k(V)$, $S \in A^l(V)$ and $T \in A^m(V)$ then

(2.1) $(R \wedge S) \wedge T = R \wedge (S \wedge T) = Alt(R \otimes S \otimes T)$ And $R \wedge (S \wedge T) = Alt\ (R \otimes Alt(S \otimes T) = Alt(R \otimes S \otimes T)$

The wedge product is associative, we can write any product $T_1 \wedge ..... \wedge T_r$ of tensor $T_i \in A^{ki}(V)$ without specifying brackets. In fact it follows by induction from that $T_1 \wedge ..... \wedge T_r = Alt(T_1 \otimes ...... \otimes T_r)$ regardless of how brackets are inserted in the wedge product in particular, it follows from $\eta_1 \wedge ..... \wedge \eta_k(v_1,....,v_k) = \frac{1}{k!}\det\left[\eta_i(v_j)_{i,j}\right]$ for all $v_1,....,v_k \in V$ and $\eta_1,....,\eta_2 \in V^*$ are viewed as 1-forms, the basis elements ζ_I are written in this fashion as $\zeta_I = \zeta_{iI} \wedge ..... \wedge \zeta_{ik}$ where $I = (i_1,....,i_k)$ is an increasing sequence form $1,....,n$ this will be our notation for ζ_I from now on . The wedge product is not commutative . Instead, it satisfies the following relation for interchange of factors.

In this defined a tensor ϕ on V is by definition a multiline V^* denoting the dual space to V , r its covariant order and s its contra variant order , assume $r \geq 0\, or\, s \geq 0$ thus ϕ assigns to each r-tupe of elements of V and s tupelo of elements of V^* a real number and if for each k ,$1 \leq k \leq r + s$ we hold every variable except the ϕ fixed the $k-th$ satisfies the linearity condition

(2.2) $$\phi(v_1,....,\alpha\, v_k + \alpha' v'_k,..) = (v_1,...,v_k) + \alpha'\phi(v'_1,.....,v_k)$$

For all $\alpha, \alpha' \in R$ and $v_k, v'_k \in V$ or V respectively for a fixed r , s we let $f_s^{\ r}(V)$ be the collection of all tensors on V of covariant order s and contra variant order r , we know that as a function from $\left(V \times .... \times V \times V^* \times ... \times V \right)$ to order R they may by added and multiplied by scalars elements R with this addition and scalar multiplication $f_s^{\ r}(V)$ is a vector space so that if $\phi_1, \phi_2 \in f_s^{\ r}(V)$ and $\alpha_1, \alpha_2 \in R$ then $\alpha_1\phi_1 + \alpha_2\phi_2$ defined in the way alluded to above that is by.

(2.3) $$(\alpha_1\phi_1 + \alpha_2\phi_2) \quad (v_1,v_2,....) = \quad \alpha_1\,\phi_1 \ (v_1,v_2,...) + \alpha_2\ \phi_2\ (v_1,v_2,...)$$

Is multiline and therefore in $f_s^{\ r}(V)$ this $f_s^{\ r}(V)$ has a natural vector space structure. In properties come naturally interims of the metric defined those spaces are known interims differential geometry as Riemannian manifolds a convector tensor on a vector V is simply a real valued $\varphi(v_1, v_2,...., v_r)$ of several vector variables $(v_1,....,v_r)$ of V the multiline number of variables is called the order of the tensor , a tensor field φ of order r on linear in each on a manifold M is an assignment to each point $p \in M$ of tensor φ_p on the vector space T_pM which satisfies a suitable regularity condition C^0, C^∞ or C^r as P on M .

Definition2.6.4

With the natural definitions of addition and multiplication by elements of R the set $f_s^{\ r}(V)$ of all tensors of order r , s on V forms a vector space of dimension n^{r+s} .

Definition2.6.5

We shall say that $\phi \in f_s^{\ r}(V)$, V a vector space is symmetric if for each $1 \leq i, j \leq r$,we $\phi(v_1, v_2,..., v_j,..., v_i,..., v_r)$ similarly if interchanging the $i-th$ and $j-th$ variables $1 \leq i, j \leq r$ changes the sign, $-\phi(v_1, v_2,..., v_j,..., v_i,..., v_r)$ then we say ϕ is skew or anti symmetric or alternating covariant tensors are often called exterior forms, a tensor field is symmetric respective alternating if it has this property at each point.

Definition 2.6.7 [Summarizing and Al-treating Transformations]

We define two liner transformations on the vector space $f_s^r(V)$, a symmertrizing mapping $f : f_s^r(V) \rightarrow f_s^r(V)$, alternating mapping.

(2.4) $$A: f_s^r(V) \rightarrow f_s^r \text{ by the formula } (f\,\phi)(v_1,....,v_r) = \frac{1}{r!}\sum_{\sigma} \sigma\phi(v_{\sigma 1},.....,v_{\sigma r})$$

And $(A\phi)(v_1,....,v_r) = \frac{1}{r!}\sum_{\sigma} \sigma\phi(v_{\sigma 1},.....,v_{\sigma r})$ the summation being over all $\sigma \in G_r$, the group of permutations of r letter it is immediate that these maps linear transformation on $f_s^r(V)$ in fact $\phi \rightarrow \phi^{\sigma}$ defined by $\phi^{\sigma}(v_r) = \phi(v_{\sigma r})$ is such that a linear transformations and any linear combination of linear transformations of a vector space is again a linear transformation.

Theorem 2.6.8

The product $f^r(V) \times f^s(V) \rightarrow f^{r+s}(V)$ just defined is bilinear associative if $w^1,....,w^n$ is abasis1 $V^* = f^1(V)$ then $w^{i(1)} \otimes,....,\otimes w^{i(r)}$ and $1 \le i_1,....,i_r \le n$ is a basis of $f^r(V)$ finally $F^* : W \rightarrow V$ is linear , then

Proof:

each statement is proved by straightforward computation to say that bilinear means that α, β are numbers $\phi_1, \phi_2 \in f^r(V)$ and $\psi \in f^r(V)$ then $(\alpha\phi_1 + \beta\phi_2) \otimes \psi = \alpha(\phi_1 \otimes \psi) + \beta(\phi_2 \otimes \psi)$ Similarly for the second variable this is checked by evaluating side on $r+s$ vectors of V in fact basis vectors suffice because of linearity associatively is similarly $(\phi \otimes \psi) \otimes \varphi = \varphi(\psi \otimes \phi)$, the defined in natural way this allows us to drop the parentheses to both $(w^{i(1)} \otimes,....,\otimes w^{i(r)})$ from a basis it is sufficient to note that if $e_1,....,e_n$ is the basis of V dual to $(w^1 \otimes .... \otimes w^n)$ then the tensor previously $\Omega^{(i1,...,ir)}$ defined is exactly $(w^{i(1)} \otimes,....,\otimes w^{i(r)})$ this follows from the two definitions.

(2.5) $$\Omega^{(i1,...,ir)}(e_{j(1)},.....,e_{j(r)}) = \begin{cases} 0 & if\ (i_1,...,i_r) \ne (j_1,....,j_r) \\ 1 & if\ (i_1,...,i_r) = (j_1,.....,j_r) \end{cases}$$

(2.6) $$(w^{i(1)} \otimes,...,\otimes w^{i(r)})(e_{j(1)},...,e_{j(r)}) = w^{i(1)}(e_{j(1)})\, w^{i(2)}(e_{j(2)}),..,w^{i(r)} = \delta_{j(1)}^{i(1)},..,\delta_{j(r)}^{i(r)}$$

which show that both tensors have the same values on any order set of r basis vectors and are thus equal finally given $F^* : W \rightarrow V$ if $w_1,....,w_{r+s}$ then

(2.7) $$\begin{aligned} F^*(\varphi \otimes \psi)(w_1,....,w_{r+s}) &= \varphi \otimes \psi(F^*(w_1),......,F^*(w_{r+s})) \\ &= \varphi(F^*(w_1),......,F^*(w_r))\,\psi(F^*(w_1),....,F^*(w_{r+s})) = (F^*\varphi) \otimes (F^*\psi)(w_1,....,w_{r+s}) \end{aligned}$$

Which proves $F^*(\varphi \otimes \psi) = (F^*\varphi) \otimes (F^*\psi)$ and completes tensor field.

Remark 2.6.9

The rule for differentiating the wedge product of a p-form α_p and q-form β_q is

(2.8) $$d(\alpha_p \wedge \beta_q) = d\alpha_p \wedge \beta_q + (-1)^p \alpha_p \wedge d\beta_q$$

Definition 2.6.10

Let $f : M \rightarrow N$ be a C^∞ map of C^∞ manifolds , then each C^∞ covariant tensor field φ on N determines a C^∞ covariant tensor field $F^*\varphi$ on M by the formula $(F^*\varphi)_p(X_{1p},....,X_{rP}) = \varphi_{F(p)}(F^*X_{1p},......,F^*X_{rp})$ the map $F^* : f^r(N) \rightarrow f^r(M)$ so defined is linear and takes symmetry alternating tensor to symmetric alternating tensors.

Lemma 2.6.11

Let $\Omega \ne 0$ be an alternating covariant tensor V of order n=dim. V and let $e_1,....,e_n$ be a basis of V then for any set of vectors $v_1,...,v_n$ with $v_i = \sum \alpha_i^j e_j$ we have, $\Omega(v_1,....,v_n) = \det|\alpha_i^j|$.

Example 2.6.12

I. Possible p-forms α_p in two dimensional space are .

(2.9)
$$\begin{aligned}\alpha_0 &= f(x,y)\\ \alpha_1 &= u(x,y)\,dx + v(x,y)\,dy\\ \alpha_2 &= \phi(x,y)\,dx\wedge dy\end{aligned}$$

The exterior derivative of line element givens the two dimensional curl times the area
$d[u(x,y)\,dx + v(x,y)\,dy\,] = (\partial_x v - \partial_y u)dx\wedge dy$.

II. the three space p-forms α_p are .

(2.10)
$$\begin{aligned}\alpha_0 &= f(x)\\ \alpha_1 &= v_1\,dx^1 + v_2\,dx^2 + v_3\,dx^3\\ \alpha_2 &= w_1\,dx^2\wedge dx^3 + w_2\,dx^3\wedge dx^1 + w_3\,dx^1\wedge dx^2\\ \alpha_3 &= \varphi(x)\,dx^1\wedge dx^2\wedge dx^3\end{aligned}$$

We see that $\alpha_1\wedge\alpha_2 = (v_1w_1 + v_2w_2 + v_1w_3)dx^1\wedge dx^2\wedge dx^3$ and

$$\begin{aligned}d\alpha_1 &= (\varepsilon_{ijk}\partial_j v_k)\frac{1}{2}\varepsilon_{ijm}dx^1\wedge dx^m\\ d\alpha_2 &= (\,\partial_1 w_1 + \partial_2 w_2 + \partial_3 w_3)dx^1\wedge dx^1\wedge dx^3\end{aligned}$$

Where ε_{ijk} is the totally anti-symmetric tensor in 3-dimensions.The isomorphism vectors tensor field we saw in the equation $\tilde{V}\equiv g(\vec{V},\cdot)\equiv g(\cdot,\vec{V})$ and $\vec{V}\equiv g^{-1}(\tilde{V},\cdot)\equiv g^{-1}(\cdot,\tilde{V})$ the link between the vector and dual vector spaces is provided by g and g^{-1} if $\vec{A}=\vec{B}$ components $A^{\mu}=B^{\mu}$ then $\tilde{A}=\tilde{B}$ components $B_{\mu}=g_{\mu\nu}B^{\nu}$ so where $A_{\mu}\equiv g_{\mu\nu}A^{\nu}$ and $B_{\mu}\equiv g_{\mu\nu}B^{\nu}$ so why do we bother one-forms when vector are sufficient the answer is that tensors may by function of both one-form and vectors , there is also an isomorphism a mongo tensors of different rank , we have just argued that the tensor space of rank (1.0) vectors and (0.1) are isomorphic , in fact all 2^{m+n} tensor space of rank $(m+n)$ with fixed $(m+n)$ are isomorphic, the metric tensor like together these spaces as exempla field by equation $T_{\mu\nu}{}^{\lambda}\equiv g(\vec{e}_{\mu},T^{k}{}_{\nu\lambda}\vec{e}_k)$ we could now use the inverse metric

(2.11)
$$T_{\mu\nu}^{\lambda}\equiv g^{-1}(\vec{e}^{\,\lambda},T_{\mu\nu k}\vec{e}^{\,k})\;\; g^{\lambda k}T_{\mu\nu k}\equiv g^{\lambda k}g_{\mu p}T_{\nu k}^{p}$$

The isomorphism of different tensor space allows us to introduce a notation that unifies them , we could effect such a unification by discarding basis vectors and one-forms only with components, in general isomorphism tensor vector $\overline{A}$ defined by

(2.12)
$$\overline{A}=\overline{A}_{\mu}\,\vec{e}^{\,-\mu}=\overline{A}_{\mu}g^{\mu\nu}\vec{e}_{\nu}\equiv\overline{A}^{\mu}\vec{e}_{\mu}$$

And $\overline{A}=\overline{A}_{\mu}\overline{e}^{\mu}$ is invariant under a change of basis because $\vec{e}^{\,\mu}$ transforms like a basis one-form .

2.7 Tensor fields

The introduced definitions allows one to introduce the tensor algebra $A_R(T_pM)$ of tensor spaces obtained by tensor products of space R and (T_pM) and $(T^*{}_pM)$. Using tensor defined on each point $p\in M$ one may define tensor fields.

Definition 2.7.1

Let M be a n-dimensional manifold . A differentiable tensor field t is an assignment $p\longrightarrow t_p$ where tensors $t_p\in A_R(T_pM)$ are of the same kind and have differentiable components with respect to all the canonical bases of $A_R(T_pM)$ given by product of bases $\left\{\frac{\partial}{\partial x^K}\;|_p\right\}k=1,\ldots,n\subset T_pM$ and $dx_p^k\,k=1,\ldots,n\subset T_p^*M$ induced by all of local coordinate system M .

In particular a differentiable vector field and a differentiable 1-form (equivalently called coveter field) are assignments of tangent vectors and 1-forms respectively as stated above.

For tensor fields the same terminology referred to tensor is used .For instance, a tensor field t which is represented in local coordinates by $t_j^i(p)\frac{\partial}{\partial x^i}\Big|_p\otimes dx^j\;|_p$ is said to of order (1,1) .

(1) It is clear that to assign on a differentiable manifold M a differentiable tensor field T (of any kind and order) it necessary and sufficient to assign a set of differentiable functions . $\left(x^1,....,x^n\right) \rightarrow T^{i1,....,im}{}_{j1,...,jk}\left(x^1,....,x^n\right)$

In every local coordinate patch (of the whole differentiable structure M or, more simply , of an atlas of M) such that they satisfy the usual rule of transformation of comports of tensors of tensors if $\left(x^1,....,x^n\right)$ and $\left(y^1,....,y^n\right)$ are the coordinates of the same point $p \in M$ in two different local charts .

$$T^{i1,...,im}{}_{j1,....,jk} \frac{\partial}{\partial x^{i1}}\bigg|_p \otimes .... \otimes \frac{\partial}{\partial x^{im}}\bigg|_p \otimes dx^{j1} \Big|_p \otimes ... \otimes dx^{jk} \Big|_p$$

(2) it is obvious that the differentiability requirement of the comports of a tensor field can be choked using the bases induced by a single atlas of local charts. It is not necessary to consider all the charts of the differentiable structure of the manifold.

(3) If X is a differentiable vector field on a differentiable manifold, M defines a derivation at each point $p \in M : if \ f \in D(M),\ X_p(f) = X^i(p)\frac{\partial}{\partial x^i}\bigg|_p$ where $\left(x^1,....,x^n\right)$ are coordinates defined about p. More generally every differentiable vector field X defines a linear mapping from $D(M)$ to $D(M)$ given by $f \rightarrow X(f)$ for every $f \in D(M)$ where $X(f) \in D(M)$ is defined as $X(f)(P) = X_p(f)$ for every $p \in M$.

(4) for (contra variant) vector field X on a differentiable manifold M , a requirement equivalent to the differentiability is the following the function $X(f) : P \rightarrow X_p(f)$, (where we use X_p as a derivation) is differentiable for all of $f \in D(M)$. Indeed if X is a differentiable contra variant vector field and if $f \in D(M)$, one has that $X(f) : P \rightarrow X_p(f)$ is a differentiable function too as having a coordinate representation .

$$X(f) \circ \phi^{-1} : \phi(U) \in \left(x^1,...,x^n\right) \rightarrow X^i\left(x^1,....,x^n\right)\frac{\partial f}{\partial x^i} \ \Big|_{(x^1,....,x^n)}$$

In every local coordinate chart (U,ϕ) and all the involved function being differentiable . Conversely $p \rightarrow X_p(f)$ defines a function in $D(M)$, $X(f)$ for every $f \in D(M)$ the components of $p \rightarrow X_p(f)$ in every local chart (U,ϕ) must be differentiable . This is because in a neighborhood of $q \in U$, $X^i(q) = X\left(f^{(1)}\right)$.

Where the function $f^{(1)} \in D(M)$ vanishes outside U and is defined as $r \rightarrow x^i(r)$, $h(r)$ in U where x^i is the i-th component of ϕ (the coordinate x^i) and h a hat function centered on q with support in U . Similarly the differentiability of a covariant vector field w is equivalent to the differentiability of each function $p \rightarrow \left\langle X_p . w_p \right\rangle$ for all differentiable vector fields X .

(5) If $f \in D(M)$ the differential of f in p , df_p is the 1-form defined by $df_p = \frac{\partial f}{\partial x^i}\bigg|_p dx^i \ \Big|_p$ in local coordinates about p .

The definition does not depend on the chosen coordinates .As a consequence , the point $p \in M$, $p \rightarrow df_p$ defines a covariant differentiable vector field denoted by df and called the differential of f .

(6) The set of contra variant differentiable vector fields on any differentiable manifold M defines a vector space with field given by R is replaced by $D(M)$, the obtained algebraic structure is not a vector space because $D(M)$ is a commutative ring with multiplicative and addictive unit elements but fails to be a field . However the incoming algebraic structure given by a vector space with the field replaced by a commutative ring with multiplicative and addictive unit elements is well know and it is called module.

III. INTEGRATION OF TENSOR FIELDS

In the previous in this section we defined the Lie derivative of tensor field along a few $a_t = \exp tX$ of a vector field X . Analogously , one can spank about an integration of tensor fields. In particular we need to recover a tensor field its known Lie derivative with respect to the vector field X .

Definition 3.1

The indefinite integral of a function f_t' with respect to the parameter t is defined as the set of all ant derivatives of f_t' along flow a_i of X symbolized by .

(3.1) $$\int f_i' \, dt = f_i + f_0$$

Where f_0 is an invariant of X i.e $X \ f_0 = 0$.

Definition 3.2

The definite integral of f_t' on a closed interval $[a,b]$ is defined by the Newton –Leibniz formula.

(3.3) $$\int_a^b f_t'\, dt = f_t \Big|_a^b = f(b) - f(a)$$

f is a tensor field, then along with the Lie differentiation one can speak about an integration of tensor fields along the flow of X. Let S and Q be smooth tensor fields of the same type on M.

Definition 3.3

A tensor fields , Q is said to be an ant derivative of S along the flow X if $Q' = l_X Q = S$ .Let Q_1 and Q_2 be tensor fields of the same ant derivative of S .then the second one is an ant derivative of S if and only if $Q_1 - Q_2 = Q_0$ where Q_0 is an invariant tensor field along the flow of X ,i.e $l_X Q_0 = 0$.

Definition 3.4

The indefinite integral of the tensor field S with respect to t is defined as the set of all ant derivatives of S along the flow a_i of X ,symbolized by .

(3.4) $$\int S_t dt = Q_t - Q_0$$

Where Q is a ant derivative of S and $l_X Q_0 = 0$ the next proposition relates the integration and the Lie differential of tensor fields.

Proposition 3.5

Let Q be an ant derivative of S along the flow a_i of X and suppose S is continuous on a closed interval $[a,b]$.Then the definite integral of S is defined by .

(3.5) $$\int S_t dt = Q_b - Q_a$$

Proof:

Let the closed interval $[a,b]$be partitioned by points $a = t_0 \le t_1 \le ..... \le t_{i-1} \le t_i \le t_{i+1} \le .... \le t_{n-1} \le t_n = b$ then the definite integral of S is defined by taking the limit of the sum.

$$\int_a^b S_t dt = \lim_{\max \Delta t_i \to 0} \sum_{i=1}^n S_{\xi i} \Delta t_i$$

Where $S_{\xi i}$ is the value of S at an arbitrary point $\xi_i \in (t_{i-1}, t_i)$ and $\Delta t_i = t_i - t_{i-1}$ is the length of the subinterval $i = 1,2,...,n$ According to the mean value theorem there is one point ξ_i in each open interval $(t_i - t_{i-1})$ such that $S\,\xi_i \Delta t_i = Q_{t\,i} - Q_{t\,i-1}$ we have $Q_b - Q_a = \sum_{i=1}^n (Q_{t\,i} - Q_{t\,i-1})$ which can be rewritten as .

(3.6) $$Q_b - Q_a = \sum_{i=1}^n S\,\xi_i \Delta t_i$$

Then taking the limit of sum in the right-hand side (3.6) as $n \to \infty$ we obtain . Let Y be differentiable vector field on M .

(3.7) $$\int_a^b [X, Y]_t\, dt = Y_a - Y_b$$

Example 3.6 (Geometrical examples)

Let us consider the linear vector field $X = -y\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ on the x,y plane the flow a_i of X is a uniform circular motion around the origin $a_i : (x, y) \to (x\cos t - y\sin t, y\cos t + x\sin t)$ the indefinite integral of a function along the flow a_i is defined by (3.1) , where $f_0 = f_0(I)$ is a function of invariant $I = x^2 + y^2$ of X form (3.3) it follows that the indefinite integral of a vector field $[X,Y]$is of the form $\int [X,Y]_t dt = Y_t + Y_0$, where Y is a differentiable vector field on the xy plane , and Y_0 is of the form $Y_0 = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$. According to condition $[X, Y_0] = (X\xi + \eta)\frac{\partial}{\partial x} + (X\eta - \xi)\frac{\partial}{\partial y} = 0$ the functions ξ and η must satisfy the system of linear ODEs .

$$\begin{cases} \xi'' + \xi' = 0 \\ \eta'' + \eta' = 0 \end{cases}$$

Where prime denotes the derivative with respect to X . Supposes two function $f = x$ and $g = y$ be given on the x, y plan .The dragging of those functions and the function $f + g = x + y$ along the flow of X are described by $f_t = x_t$, $g_t = y_t$ and $(f + g)_t = (x + y)_t$ respectively . Let us calculate the corresponding definite integrals on the closed interval $[a,b] = \left[0, \frac{\pi}{2}\right]$.

By (3.2) we have.

$$\int_0^{\frac{\pi}{2}} f_t' dt = \int_0^{\frac{\pi}{2}} x_t' dt = x_t \Big|_0^{\frac{\pi}{2}} = -x - y$$

$$\int_0^{\frac{\pi}{2}} g_t dt = \int_0^{\frac{\pi}{2}} y_t dt = y_t \Big|_0^{\frac{\pi}{2}} = x - y$$

$$\int_0^{\frac{\pi}{2}} (f+g)_t' dt = \int_0^{\frac{\pi}{2}} (x_t - y_t)' \, dt = (x_t + y_t)_0^{\frac{\pi}{2}} = -2y$$

Consider the vector field $Y = \frac{\partial}{\partial y}$ the Lie derivatives of Y with respect to X is described the vector – function $Y' = [X.Y] = \frac{\partial}{\partial x}$, $Y'' = [X.Y'] = -\frac{\partial}{\partial y} = -Y$ thus we have $Y'' + Y' = 0$ and the dragging of Y along the flow of X is described by the vector-function $Y_t = Ta_t Y = \sin t \frac{\partial}{\partial x} + \cos t \frac{\partial}{\partial y}$ then using we obtain the definite integral of field $Y' = \frac{\partial}{\partial x}$ on the closed interval $\left[0, \frac{\pi}{2}\right]$

$$\int_0^{\frac{\pi}{2}} \left(\frac{\partial}{\partial x}\right)_t dt = Y_t \Big|_0^{\frac{\pi}{2}} = \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \ , \ \left(\frac{\partial}{\partial x}\right)_t = \cos t \frac{\partial}{\partial x} - \sin t \frac{\partial}{\partial y}$$

The **figure (1**) illustrate the meaning of the definite integral of a vector field on the .

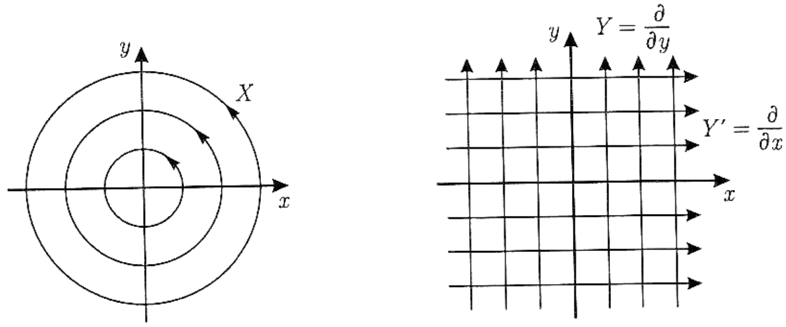

Figure (1) : the flow of X is the uniform circular motion around the origin in the counterclockwise direction . The Lie derivative of $Y = \frac{\partial}{\partial y}$ (south wind) with respect to X is the field $Y' = \frac{\partial}{\partial x}$ (west wind) .

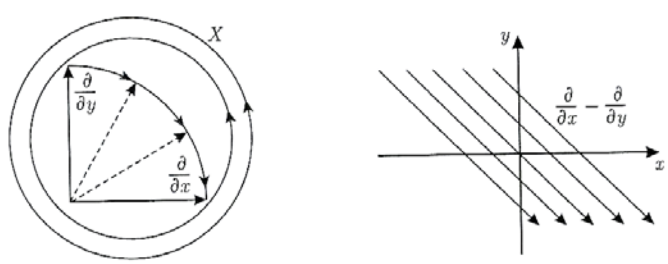

Figure (2) : The field Y is rotated in moving frame according to the law $Y_t = Y \cos t + Y' \sin t$ (the wind changes own direction rotating clockwise) . The calculating of definite integral $\int_0^{\frac{\pi}{2}} [X.Y] dt$ yields the field $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ (north-west wind)

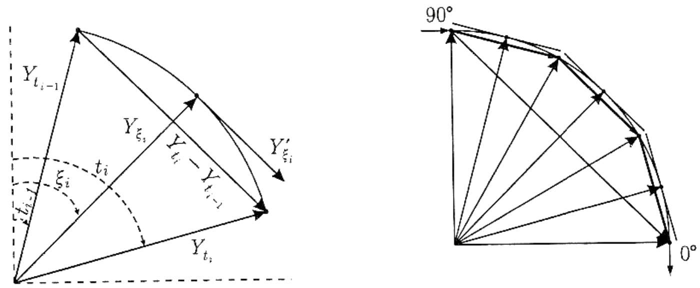

Figure (3): the summands for the integral sum are defined by the mean value theorem taking the limit of the integral sum. We obtain the closing line to the hodograph of Y_i, the hodograph is the velocity as function of time . The hodograph of the vector-function Y_i has the same trajectory as X but with opposite direction . The integral sum $\Sigma'_{\xi_t} Y'_{\xi_t} \Delta t_t$ is a broken line to the hodograph and the integral $\frac{\partial}{\partial x} - \frac{\partial}{\partial y}$ is a straight line closing this broken , see Figure (3) .

Example 3.7

Let three vector fields $X = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$, $Y = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}$, $Z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$

Be given in space in the space xyz . The flows of Y, X and Z are rotation about three axis xyz respectively . Let us consider the dragging of Y along the flow of X the dragging of Z along the flow of Y and the dragging of X along the flow of Z :

$$Y' = [X, Y] = Z \ , Y'' + Y = 0 \Rightarrow Y_t = Y\cos t + Z\sin t$$

$$Z' = [Y, Z] = X \ , Z'' + Z = 0 \Rightarrow Z_t = Z\cos t + X\sin t$$

$$X' = [Z, Y] = Y, X'' + X = 0 \Rightarrow X_t = X\cos t + Y\sin t$$

Let us calculate the integrals of Y, X and Z on a closed interval $[a, b]$.

$$\int_a^b Z_t dt = \int_a^b [X, Y]_t dt = Y_b - Y_a = 2\sin\frac{a-b}{2}\left(Y\sin\frac{a+b}{2} - Z\cos\frac{a+b}{2}\right)$$

$$\int_a^b X_t dt = \int_a^b [Y, Z]_t dt = Z_b - Z_a = 2\sin\frac{a-b}{2}\left(Z\sin\frac{a+b}{2} - X\cos\frac{a+b}{2}\right)$$

$$\int_a^b Y_t dt = \int_a^b [Z, X]_t dt = X_b \_ X_a = 2\sin\frac{a-b}{2}\left(X\sin\frac{a+b}{2} - Z\cos\frac{a+b}{2}\right)$$

Taking $a = 0$ and $b = \frac{\pi}{2}$ we obtain three vector fields .

(3.8) $$\int_0^{\frac{\pi}{2}} Z_t dt = (y - z)\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}$$

(3.9) $$\int_0^{\frac{\pi}{2}} X_t dt = -y\frac{\partial}{\partial x} - (x + z)\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$$

(3.10) $$\int_0^{\frac{\pi}{2}} Y_t dt = y\frac{\partial}{\partial x} - (x + z)\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$$

The flow of the field (3.8) is .

$$(x, y, z) \rightarrow \{x_t = x\cos\sqrt{2t} + (y + z)\frac{\sin\sqrt{2t}}{\sqrt{2}}$$

$$(x, y, z) \rightarrow \{y_t = y - x\frac{\sin\sqrt{2t}}{\sqrt{2}} - (y + z)\frac{1 - \cos\sqrt{2t}}{2}$$

$$(x, y, z) \rightarrow \{z_t = z - x\frac{\sin\sqrt{2t}}{\sqrt{2}} - (y + z)\frac{1 - \cos\sqrt{2t}}{2}$$

From the equalities $y_t - z_t = y - z$ and $2x_t^2 + (y+z)^2 = 2x^2 + (y+z)^2$ we obtain two invariants .

(3.11) $$I_1 = 2x^2 + (y+z)^2 \quad , \; I_2 = y - z$$

It means that the level surfaces of the trajectories of the field (3.9) are elliptic cylinders with axis of rotation $y+z=0 \; , x=0$. The trajectories are ellipse on the intersections of cylinders $I_1 = c \geq 0$ with plane $I_2 = c \geq 0$ perpendicular to the axis of rotation the flow of the field (3.9) is .

$$(x,y,z) \to \{x_t = x - y\frac{\sin\sqrt{2t}}{\sqrt{2}} - (x+z)\frac{1-\cos\sqrt{2t}}{\sqrt{2}}$$

$$(x,y,z) \to \{y_t = y\cos\sqrt{2t} + (x+z)\frac{\sin\sqrt{2t}}{\sqrt{2}}$$

$$(x,y,z) \to \{z_t = z - y\frac{\sin\sqrt{2t}}{\sqrt{2}} - (x+z)\frac{1-\cos\sqrt{2t}}{2}$$

And the invariants are $I_1 = 2x^2 + (y+z)^2 \quad , I_2 = y - z$ the level surface the trajectories of the field (3.9) are elliptic with axis of rotation $y+z=0 \; , x=0$ the trajectories are ellipses on the intersection of the cylinders $I_1 = c \geq 0$ with planes $I_2 = c \geq 0$ perpendicular to the of rotation from $\int_0^{\frac{\pi}{2}} Y_t dt = -\int_0^{\frac{\pi}{2}} X_t dt$ it follows that the flow and invariants of the fields (3.9) and (3.10) are the sume, but the trajectories of these fields are opposite directed.

IV. CONCLUSION

(a) The introduced definitions allows one to introduce the tensor algebra $A_R(T_pM)$ of tensor spaces obtained by tensor products of space R and (T_pM) and $(T^*{}_pM)$. Using tensor defined on each point $p \in M$ one may define tensor fields.

(b) the tensor product $(S,T) \to S \otimes T$ form $T^k(V) \times T^l(V) \to T^{k+1}(V)$, there is a construction of a product $A^k(V) \times A^l(V) \to A^{k+1}$ since tensor products of alternating tensors are not alternating.

(c) The definite integral of f_t' on a closed interval $[a,b]$ is defined by the Newton –Leibniz formula.

(d) $\int_a^b f_t' \; dt = f_t \Big|_a^b = f(b) - f(a)$

(e) f is a tensor field, then along with the Lie differentiation one can speak about an integration of tensor fields along the flow of X . S and Q be smooth tensor fields of the same type on M .

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