On Fixed Point Theorems and their Applications (I)

N. H. Mohammed [*], M.A.Bashir [**]

[*] Math Dept College of Engineering-Sinnar University-Sudan
[**] Math Dept El-Neenlain University-Sudan

I. Introduction

The fixed point theorems concern maps f of a set X into itself that, under certain conditions, admit a fixed point, that is, a point $x \in X$ such that $f(x) = x$. Which the knowledge of the existence of fixed points has relevant applications in many branches of analysis and topology. Then let us show for instance the following simple but indicative example.

Suppose we given a system of n equations in n unknowns of the form $g_j(x_1, \dots, x_n) = 0, j = 1, \dots, n$ where the g_j are continuous real-valued functions of the real variables x_j. Let $h_j(x_1, \dots, x_n) = g_j(x_1, \dots, x_n) + x_j$, and for any point $x = (x_1, \dots, x_n)$ define $h(x) = (h_1(x), \dots, h_n(x))$. Assume that h has a fixed point $\tilde{x} \in R^n$. Then we see that $\bar{x}$ is a solution to the system of equations..

II. The Banach contraction principle

Definition

Suppose that X is a metric space equipped with a distance d. A map $f : X \to X$ is said to be Lipschitz continuous if there is $\lambda \geq 0$ such that $d(f(x_1), f(x)) \geq d\, \lambda\, (x_1, x_2),\ \forall x_1, x_2 \in X$. The smallest λ for which the above inequality holds is the Lipschitz constant of f. If $\lambda \leq 1$ f is said to be non-expansive, if $\lambda < 1$ f is said to be a contraction.

2.1 Theorem [Banach]

Suppose that f is a contraction on a complete metric space X. Then f has a unique fixed point $\bar{x} \in X$.

Proof:

if $x_1, x_2 \in X$ are fixed points of f, then
$d(x_1, x_2) = d(f(x_1), f(x_2)) \leq \lambda\ d(x_1, x_2)$
which imply $x_1 = x_2$. Choose now any $x_0 \in X$, and define the iterate sequence $x_{n+1} = f(x_n)$. By induction on n,
$d(x_{n+1}, x_n) \leq \lambda^n d(f(x_0), x_0)$.
if $n \in N$ and $m \geq 1$,

$$d(x_{n+m}, x_n) \leq d(x_{n+m}, x_{n+m-1}) + \cdots + d(x_{n+1}, x)$$
$$\leq (\lambda^{n+m} + \cdots + \lambda^n)\, d(f(x_0), x_0) \qquad (1)$$

x_n limit $\bar{x} \in X$. Hence x_n is a Cauchy sequence, and admits a limit $\bar{x} \in$ X, for X is complete. Since f is continuous, then we have $f(\bar{x}) = \lim_n f(x_n) = \lim_n x_{n+1} = \bar{x}$.

***Remark*:**

Let $m \to \infty$ in (1) we find the relation

$d(x_n, \bar{x}) \leq \frac{\lambda^n}{1-\lambda} d(f(x_0), x_0)$ which provides a control on the convergence rate of x_n to the fixed point $\bar{x}$. The completeness of X plays here a crucial role. Indeed, contractions on incomplete metric spaces may fail to have fixed points.

(2.2)Example :

Suppose that $X = (0, 1]$ with the usual distance. Define $f : X \to X$ as $f(x) = x/2$.

(2.3) Corollary

Let X be a complete metric space and Y be a topological space. Let $f : X \times Y \to X$ is a continuous function. Assume that f is a contraction on X uniformly in Y, that is,

$d(f(x_1, y), f(x_2, y)) \leq \lambda\, d(x_1, x_2),\ \forall x_1, x_2 \in X,\ \forall y \in Y$
for some $\lambda < 1$. Then, for every fixed $y \in Y$, the map

$x \mapsto f(x, y)$ has a unique fixed point $\varphi(y)$. Moreover, the function $y \mapsto \varphi(y)$ is continuous from Y to X. Notice that

if $f : X \times Y \to X$ is continuous on Y and is a contraction on X uniformly in Y, then f is in fact continuous on $X \times Y$.

Proof:

In light of Theorem 1.3, we only have to prove the continuity of φ. For $y, y_0 \in Y$, *we have*
$d(\varphi(y), \varphi(y_0)) = d(f(\varphi(y), y), f(\varphi(y_0), y_0))$
$\leq d(f(\varphi(y), y), f(\varphi(y_0), y)) + d(f(\varphi(y_0), y), f(\varphi(y_0), y_0))$
$\leq \lambda\, d(\varphi(y), \varphi(y_0)) + d(f(\varphi(y_0), y), f(\varphi(y_0), y_0))$
which implies

$$d(\varphi(y),\varphi(y_0)) \leq \frac{1}{1-\lambda} d(f\varphi(y_0), y), f(\varphi(y_0), y_0)).$$

Since the above right-hand side goes to zero as $y \to y_0$, we have L >0 the desired continuity.

Remark

If in addition Y is a metric space and f is Lipschitz continuous in Y , uniformly with respect to X, with Lipschitz constant $L \geq 0$, then the function $y \mapsto \varphi(y)$ is Lipschitz continuous with Lipschitz constant less than or equal to $L/(1 - \lambda)$.Theorem 2.1 gives a sufficient condition for f in order to have a unique fixed point.

(2.4)Example

Consider the map

$$g(x) = \begin{cases} \frac{1}{2} + 2x & x \in [0, \frac{1}{4}] \\ \frac{1}{2} & x \in [\frac{1}{4}, 1] \end{cases}$$

mapping [0, 1] onto itself. Then g is not even continuous, but it has a unique fixed point ($x = 1/2$). The next corollary takes into account the above situation, and provides existence and uniqueness of a fixed point under more general conditions.

(2.5)Definition:

For $f : X \to X$ and n ∈ N, we denote by f^n the n[th] -iterate of f, namely, $f \circ, \cdots \circ f$ n-times (f^0 is the identity map).

(2.6) Corollary :

Suppose that X is a complete metric space and let $f : X \to X$. If f^n is a contraction, for some $n \geq 1$, then f has a unique fixed point $\bar{x} \in$ X.

proof :

Let $\bar{x}$ be the unique fixed point of f^n, given by Theorem 1.3. Then $fn(f(\bar{x})) = f(f^n(\bar{x})) = f(\bar{x})$, which implies $f(\bar{x}) = \bar{x}$. Since a fixed point of f is clearly a fixed point of f^n, we have uniqueness as well. Notice that in the example $g^2(x) \equiv 1/2$.

III. Further Extensions of the Contraction Principle

There is in the literature a great number of generalizations of Theorem 2.1 Here we point out some results.

Theorem [Boyd-Wong] :Let X be a complete metric space, and let $f : X \to X$. Assume there exists a right-continuous function $\varphi : [0,1) \to [0,1)$ such that $\varphi(r) < r$ if $r > 0$, and $d(f(x_1), f(x_2)) < \varphi(d(x_1, x_2))$, $\forall x_1, x_2 \in X$. Then f has a unique fixed point $\bar{x} \in$ X. Moreover, for any $x_0 \in X$ the sequence $f^n(x_0)$ converges to $\bar{x}$. Clearly, Theorem 2.1 is a particular case of this result, for $\varphi(r) = \lambda r$.

Proof:

If $x_1, x_2 \in X$ are fixed points of f, then

$d(x_1, x_2) = d(f(x_1), f(x_2)) < \varphi(d(x_1, x_2))$ fixed point theorems

so $x_1 = x_2$. To prove the existence, fix any $x_0 \in X$, and define the iterate sequence $x_{n+1} = f(x_n)$. We show that x_ε is a Cauchy sequence, and the desired conclusion follows arguing like in the proof of Theorem 2.1. For $n \geq 1$, define the positive sequence $a_n = d(x_n, x_{n-1})$. It is clear that $a_{n+1} < \psi(a_n) < a_n$; therefore a_n converges monotonically to some $a \geq 0$. From the right-continuity of ψ, we get $a \leq \psi(a)$, which entails a = 0. If x_n is not a Cauchy sequence, there is $\varepsilon > 0$ and integers $m_k \geq n_k \geq k$ for every $k \geq 1$ such that

$d_k := d(x_{mk}, x_{nk}) \geq \varepsilon, \ \forall k \geq 1$. In addition, upon choosing the smallest possible m_k, we may assume that $d(x_{mk-1}, x_{nk}) < \varepsilon$ for k big enough (here we use the fact that $a_n \to 0$). Therefore, for k big enough,

$\varepsilon \leq d_k \leq d(x_{mk}, x_{mk-1}) + d(x_{mk-1}, x_{nk}) < a_{mk} + \varepsilon$

implying that $d_k \to \varepsilon$ from above as $k \to \infty$. Moreover,

$d_k < d_{k+1} + a_{mk+1} + a_{nk+1} \leq \varphi(d_k) + a_{mk+1} + a_{nk+1}$

and taking the limit as $k \to \infty$ we obtain the relation $\varepsilon \leq \varphi(\varepsilon)$, which has to be false since $\varepsilon > 0$.

(3.1)Theorem [Caristi] :

Let X be a complete metric space, and let $f : X \to X$. Assume there exists a lower semicontinuous function

$\psi : X \to [0,1)$ such that $d(x, f(x)) \leq \psi(x) - \psi(f(x))$, $\forall x \in X$. Then f has (at least) a fixed point in X. Again, Theorem 2.1 is a particular case, obtained for $\psi(x) = d(x, f(x)) / (1-\lambda)$. Notice that f need not be continuous.

Proof:

We introduce a partial ordering on X, setting $x \leq y$ if and only if $d(x, y) \leq \psi(x) - \psi(y)$. Let $;\phi \neq X_0 \subset X$ be totally ordered, and consider a sequence $x_n \in X_0$ such that $\psi(x_n)$ is decreasing to $\alpha := \inf\{\psi(x) : x \in X_0\}$. If n ∈ N and m ≥ 1,

$$d(x_{n+m}, x_n) \leq \sum_{i=0}^{m-1} d(x_{n+m+i+1}, x_{n+i})$$
$$\leq \sum_{i=0}^{m-i} \psi(x_{n+I} - \psi(x_i), \psi(x_{n+i+1}))$$
$$= \psi(x_n) - \psi(x_{n+m}).$$

Hence x_n is a Cauchy sequence, and admits a limit $x_n \in X$, for X is complete. Since can only jump downwards (being lower semicontinuous), we also have $\psi(x_*) = \alpha$. If $x \in X_0$ and $d(x, x_*) > 0$, then it must be $x \preceq x_n$ for large n. Indeed, $\lim_n \psi(x_n) = \psi(x_*) \leq \psi(x)$. We conclude that x_* is an upper bound for X_0, and by the Zorn lemma there exists a maximal element $\bar{x}$. On the other hand, $\bar{x} \leq f(\bar{x})$, thus the maximality of $\bar{x}$ forces the equality $\bar{x} = f(\bar{x})$. If we assume the continuity of f, we obtain a slightly stronger result, even relaxing the continuity hypothesis on ψ.

(3.2)Theorem:

Let X be a complete metric space, and let $f : X \to X$ be a continuous map. Assume there exists a function $: X \to [0,\infty)$ such that $d(x, f(x)) \leq \psi(x) - \psi(f(x))$, $\forall x \in X$.

Then f has a fixed point in X. Moreover, for any $x_0 \in X$ the sequence $f^n(x_0)$ converges to a fixed point of f.

Proof :

Let $x_0 \in X$. Due the above condition, the sequence $\psi(f^n(x_0))$ is decreasing, and thus convergent. Reasoning as in the proof of the Caristi theorem, we get that $f^n(x_0)$ admits a limit $\bar{x} \in$ X, for X is complete. The continuity of f then entails $f(\bar{x}) = \lim_n f(f^n(x_0)) = \bar{x}$. We conclude with the following extension of Theorem 1.3, that we state without proof.

(3,3)Theorem [$\overline{C}$iri$\overline{c}$] :

Let X be a complete metric space, and let $f : X \to X$ be such That $d(f(x_1), f(x_2)) \leq \lambda \max\{d(x_1, x_2), d(x_1, f(x_1)), d(x_2, f(x_2)), d(x_1, f(x_2)), d(x_2, f(x_1))\}$ for some $\lambda < 1$ and every $x_1, x_2 \in X$. Then f has a unique fixed point $\bar{x} \in X$. Moreover, $d(f^n(x_0), \bar{x}) = O(\lambda^n)$ for any $x_0 \in X$. Also in this case f need not be continuous. However, it is easy to check that it is continuous at the fixed point. The function g of the former example fulfills the hypotheses of the theorem with $\lambda = 2/3$.

IV. Weak contractions

We now dwell on the case of maps on a metric space which are contractive without being contractions.

(4.1)Definition :

Let X be a metric space with a distance d. A map $f : X \to X$ is a weak contraction if

$d(f(x_1), f(x_2)) < d(x_1, x_2), \ \forall x_1 \neq x_2 \in X.$

Being a weak contraction is not in general a sufficient condition for f in order to have a fixed point, as it is shown in the following simple example.

(4.2)Example:

Consider the complete metric space $X = [1,+\infty)$, and let $f : X \to X$ be defined as $f(x) = x+1/x$. It is easy to see that f is a weak contraction with no fixed points. Nonetheless, the condition turns out to be sufficient when X is compact.

(4.3)Theorem :

Let f be a weak contraction on a compact metric space X. Then f has a unique fixed point $\bar{x} \in X$. Moreover, for any $x_0 \in X$ the sequence $f^n(x_0)$ converges to $\bar{x}$.

Proof:

The uniqueness argument goes exactly as in the proof of Theorem 2.1. From the compactness of X, the continuous function $x \mapsto d(x, f(x))$ attains its minimum at some $\bar{x} \in X$. If $\bar{x} \neq f(\bar{x})$, we get

$$d(\bar{x}, f(\bar{x})) = \min_{x\in X} d(x, f(x)) \leq d(f(\bar{x}), f(f(\bar{x}))) < d(\bar{x}, f(\bar{x}))$$

which is impossible. Thus $\bar{x}$ is the unique fixed point of f (and so of f^n for all $n \geq 2$). Let now $x_0 \neq \bar{x}$ be given, and define $d_n = d(f^n(x_0), \bar{x})$. Observe that

$d_{n+1} = d(f^{n+1}(x_0), f(\bar{x})) < d(f^n(x_0), \bar{x}) = d_n$.

Hence d_n is strictly decreasing, and admits a limit $r \geq 0$. Let now $f^{n_k}(x_0)$ be a subsequence of $f^n(x_0)$ converging to some $z \in X$. Then $r = d(z,\bar{x}) = \lim_{k\to\infty} d_{nk} = \lim_{k\to\infty} d_{n_{k+1}}$

$= \lim_{k\to\infty} d(f(f^n{}_k(x_0)), \bar{x}) = d(f(z), \bar{x})$. But if $z \neq \bar{x}$, then $d(f(z), \bar{x}) = d(f(z), f(\bar{x})) < d(z, \bar{x})$. Therefore any convergent subsequence of $f^n(x_0)$ has limit $\bar{x}$, which, along with the compactness of X, implies that $f^n(x_0)$ converges to $\bar{x}$. Obviously, we can relax the compactness of X by requiring that $\overline{f(x)}$ be compact (just applying the theorem on the restriction of f on $\overline{f(x)}$ Arguing like in Corollary 2.3 , it is also immediate to prove the following

(4.4)Corollary:

Suppose that X is a compact metric space and let $f : X \to X$. If f^n is a weak contraction, for some $n \geq 1$, then f has a unique fixed point $\bar{x} \in X$.

(4.5) A converse to the contraction principle :

Assume we are given a set X and a map $f : X \to X$. We are interested to find a metric d on X such that (X, d) is a complete metric space and f is a contraction on X. Clearly, in light of Theorem 2.1,a necessary condition is that each iterate f^n has a unique fixed point. Surprisingly enough, the condition turns out to be sufficient as well. sequences of maps and fixed points .

(4.6)Theorem [Bessaga] :

Suppose that X is an arbitrary set, and let $f : X \to X$ be a map such that f_n has a unique fixed point $\check{\bar{x}} \in X$ for every $n \geq 1$. Then for every $\varepsilon \in (0, 1)$, there is a metric $d = d\varepsilon$ *on* X that makes X a complete metric space, and f is a contraction on X with Lipschitz constant equal to ε.

proof :

suppose $\varepsilon \in (0, 1)$ and let Z be the subset of X consisting of all elements z such that $f^n(z) = \bar{x}$ for some $n \in N$. We define the following equivalence relation on $X \setminus Z$: we say that x~ y if and only if $f^n(x) = f^m(y)$ for some $n,m \in N$. Notice that if $f^n(x) = f^m(y)$ and $f^{\tilde{n}}(x) = f^{m'}(y)$ then $f^{n+m'}(x) = f^{m+n'}(x)$.

But since $x \neq Z$, this yields $n + m' = m + n'$, that is, $n - m = n' - m'$. At this point, by means of the axiom of choice, we select an element from each equivalence class. We now proceed defining the distance of $\bar{x}$ from a generic $x \in X$ by setting $d(\bar{x}, \bar{x}) = 0$, $d(x, \bar{x}) = \varepsilon^{-n}$ if $x \in Z$ with $x \neq \bar{x}$, where $n = \min\{m \in N : f^m(x) = \bar{x}\}$, and $d(x, \bar{x}) = \varepsilon^{n-m}$ if $x \neq Z$, where $n,m \in N$ are such that $f^n(\hat{x}) = f^m(x)$, $\hat{\bar{x}}$ being the selected representative of the equivalence class [x]. The definition is unambiguous, due to the above discussion. Finally ,for any $x, y \in X,$ we set d(x, y)

$$= \begin{cases} d(x,\bar{x}) + d(y,\bar{x}) & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

It is straightforward to verify that d is a metric. To see that d is complete, observe that the only Cauchy sequences which do not converge to $\bar{x}$ are ultimately constant. We are left to show that f is a contraction with Lipschitz constant equal to ε. Let x ∈ X,

$x \neq x$. If $x \in Z$ we have

$d(f(x), f(\bar{x})) = d(f(x), \bar{x}) \leq \varepsilon^{-n} = \varepsilon\, \varepsilon-(n+1) = \varepsilon d(x, \bar{x}).$

If $x \neq Z$ we have

$d(f(x), f(\bar{x})) = d(f(x), \bar{x}) = \varepsilon^{n-m} - \varepsilon\varepsilon^{n-(m+1)}) = \varepsilon d(x, \bar{x})$

since x ~ f(x). The thesis follows directly from the definition of the distance.

V. Sequences of Maps and Fixed Points

Suppose that (X, d) be a complete metric space. We consider the problem of convergence of fixed points for a sequence of $f_n : X \to X$. Corollary 2.3 will be implicitly used in the statements of the next two theorems.

(5.1) Theorem

Assume that each f_n has at least a fixed point $x_n = f_n(x_n)$. Let $f : X \to X$ be a uniformly continuous map such that f^m is a contraction for some $m \geq 1$. If f_n converges uniformly to f, then x_n converges to $\bar{x} = f(\bar{x})$.

Proof: We assume that f is a contraction (i.e., m = 1).

Let λ < 1 be the Lipschitz constant of f. Given ε > 0, choose $n_0 = n_0(\varepsilon)$ such that $d(f_n(x), f(x)) < \varepsilon(1-\lambda)$Then, for $n \geq n_0$,

$d(x_n, \bar{x}) = d(f_n(x_n), f(\bar{x}))$

$\leq d(f_n(x_n), f(x_n)) + d(f(x_n), f(\bar{x}))$

$\leq \varepsilon(1-\lambda) + \lambda d(x_n, \bar{x}).$

Therefore $d(x_n, \bar{x}) \leq \varepsilon$, which proves the convergence.

To prove the general case we observe that if $d(f^m(x), f^m(y)) \leq \lambda^m d(x, y)$ for some λ < 1, we can define a new metric d_0 on X equivalent to d by letting

$$d^0(x, y) = \sum_{k=0}^{m-1} \frac{1}{\lambda^k} d(f^k(x), f^k(y)).$$

Moreover, since f is uniformly continuous, f_n converges uniformly to f also with respect to d_0. Finally, f is a contraction with respect to d_0.

$$d_0(f(x), f(y)) = \sum_{k=0}^{m-1} \frac{1}{\lambda^k} d(f^{k+1}(x), f^{k+1}(y))$$

$$= \lambda \sum_{k=0}^{m-1} \frac{1}{\lambda^k} d(f^k(x), f^k(y)) + \frac{1}{\lambda^{m+1}} d(f^m(x), f^m(y)) \leq \lambda \sum_{k=0}^{m-1} \frac{1}{\lambda^k} d(f^k(x), f^k(y)) = \lambda d_0(x, y).$$

So the problem is reduced to the previous case m = 1.The next result refers to a special class of complete metric spaces.

(5.2)Theorem:

Suppose that X is locally compact. Assume that for each $n \in \mathbb{N}$ there is $m\, n \geq 1$ such that f_n^{mn} is a contraction. Suppose that $f : X \to X$ be a map such that f_m is a contraction for some $m \geq 1$ equicontinuous family, then $x_n = f^n(x_n)$ converges to $\bar{x} = f(\bar{x})$.

Proof:

Suppose trhat ε = 0 is sufficiently small such that $K(\bar{x}, \varepsilon) := \{x \in X : d(x, \bar{x}) \leq \varepsilon \subset X$ is compact. As a byproduct of the Ascoli theorem, f_n converges to f uniformly on $K(\bar{x},\varepsilon)$, since it is equicontinuous and pointwise convergent. Let $N_0 = n_0(\varepsilon)$ such that $d(f^{m\,n}(x), f^m(x)) \leq \varepsilon(1-\lambda), \forall n \geq n_0$, $\forall x \in K(\bar{x}, \varepsilon)$ *fixed* points of non-expansive maps where λ< 1 is the Lipschitz constant of f^m. Then, for $n \geq n_0$ and $x \in K(\bar{x}, \varepsilon)$ we have $d(f^{m\,n}(x), \bar{x}) = d(f^{mn}(x), f^m(\bar{x}))$

$\leq d(f^{mn}(x), f^m(x)) + d(f^m(x), f^m(\bar{x}))$

$\leq \varepsilon(1-\lambda\_) + \lambda d(x, \bar{x}) \leq \varepsilon.$

Hence $f_m^m(K(\bar{x}, \varepsilon)) \subset K(\bar{x}, \varepsilon)$ for all $n \geq n_0$. Since the maps f_n^{mn} are contractions, it follows that, for $n \geq n_0$, the fixed points x_n of f_n belong to $K(\bar{x}, \varepsilon)$,that is, $d(x_n, \bar{x}) \leq \varepsilon$.

VI. Fixed Points of Non-Expansive Maps

Suppose that X is a Banach space, C ⊂ X nonvoid, closed, bounded and convex, and then $f : C \to C$ be a non-expansive map. The problem is whether f admits a fixed point in C. The answer, is false ,in general.

(6.1)Example:

Suppose that $X = c^0$ with the supremum norm.

Letting $C = \bar{B}_X(0, 1)$, the map f : C → C defined by

$f(x) = (1, x_0, x_1, \ldots)$, for $x = (x_0, x_1, x_2, \ldots) \in C$ is non-expansive but clearly admits no fixed points in C. Things are quite different in uniformly convex Banach spaces.

(6.2)Theorem [Browder-Kirk]:

Suppose that X is a uniformly convex Banach space and $C \subset X$ be nonvoid, closed, bounded and convex. *If* $f : C \to C$ *is* a non-expansive map, then f has a fixed point in C. In the particular case when X is a Hilbert space .

Proof:

Suppose that $x_* \in C$ is a fixed point, and consider a sequence $r_n \in (0, 1)$ converging to 1. For each $n \in N$, define a map $f_n : C \to C$ as $f_n(x) = r_n(x) + (1 - r_n)x_*$ .Notice that f^n is a contractions on C, hence there is a unique $x_n \in C$ such that $f_n(x_n) = x_n$. Since C is weakly compact, x_n has a subsequence (still denoted by x_n) weakly convergent to some $\bar{x} \in C$. We shall prove that $\bar{x}$ is a fixed point of f. Notice first that

$$\lim_{n \to \infty} \|f(\bar{x}) - x_n\|^2 - \|\bar{x} - x_n\|^2 = \|f(\bar{x}) - \bar{x}\|^2$$

Since f is non-expansive we have

$$\le \|\bar{x} - x\| + \|f(x_n - x_n)\|$$

$$= \|\bar{x} - x_n\| + (1 - r_n)\|f_n(x) - x\|$$

But $r_n \to 1$ as $n \to \infty$ and C is bounded, then we conclude that

$$\lim_{n \to \infty} \sup \|f(\bar{x}) - x_n\|^2 - \|\bar{x} - x_n\|^2 \le 0,$$

which yields that the equality $f(\bar{x}) = \bar{x}$.

(6.3)Proposition:

In Theorem 3.1, the set *F* of fixed points of *f* is closed and convex.

proof :

The first assertion is trivial. Assume then $x_0, x_1 \in F$, with $x_0 \neq x_1$, and denote $x_t = (1 - t)x_0 + tx_1$, with $t \in (0, 1)$. We have

$$\|f(x_t) - x_0\| = \|f(x_t) - f(x_0)\|$$

$$\le \|x_t - x_0\| + \|x_1 - x_0\|$$

$$\|f(x_1) - x_1\| = \|f(x_t) - f(x_1)\|$$

$$\le \|x_t - x_1\| = (1-t)\|x_1 - x_0\|$$

that imply the equalities

$$\|f(x_t - x_0)\| = t\|x_t - x_0\|$$

$$\|f(x_t - x_1)\| = (1 - t)\|x_1 - x_0\|.$$

The proof is completed if we show that $f(x_t) = (1-t)x_0 + tx_1$. This follows from a general fact about uniform convexity.

VII. Integral Equations

Let $a, b \in R$ with $a < b$. Let $(x, y) \mapsto K(x,y)$ be a measurable function on $\{a < x < b; a < y < b\}$.

7.1 .Theorem.

Suppose that $\int_a^b \int_a^b | k(x,y)|^2 dx\, dy) < \infty$

and $g \in L^2([a, b])$. Then the integral equation

$$f(x) = g(x) + \mu \int_a^b (x,y) f(y) dy$$

has a unique solution for $|\mu| \le \| k \|^{-1}_{KL^2(|a,b| \times |a,b|)}$

Proof: We claim that the function h defined by

$$h(x) := g(x) + \mu \int_a^b k(x,y) f(y) dy$$

where $f \in L^2([a, b])$, lies in $L\in([a; b])$. By linearity, the triangle inequality, and our hypothesis that $g \in L\in([a, b])$,

we only need to show that

$\Psi(x) := \sum_a^b k(x,y) f(y) dy \in L^2([a, b])$. By Fubini's theorem and Hölder's inequality,

$$\int_a^b |\Psi(x)|^2) dx \le ($$

$$\int_a^b \int_a^b k(x,y) f(y) dy|)^2$$

$$\int_a^b | k(x,y|^2 dy) (\int_a^b | f(y)|^2 dy) dx$$

$$= (\int_a^b \int_a^b | k(x,y)|^2 dy dx)(\int_a^b | f(y)|^2 dy) < \infty$$

Define a mapping $T : L^2([a, b]) \to L^2([a, b])$ by $Tf := h$, where the metric d is the standard L^2 metric. For

$f_1; f_2 \in L^2([a; b])$, we have by Hölder's inequality that

$$d(Tf_1; \quad Tf_2) = |\mu| / (\int_a^b | \int_a^b k(x,y) \quad f_1(y) dy|^2 dx)^{\frac{1}{2}}$$

$\trianglelefteq \mu$ | (

$$\int_a^b \int_a^b | k(x,y) |^2 dy \left(\int_a^b | f_1(y) - f_2(y)|^2 dy\right) dx)^{\frac{1}{2}}$$

$$\trianglelefteq \mu| \int_a^b \int_a^b | k(x,y) |^2 dydx)^{\frac{1}{2}} d(f_1 . f_2)$$

If $|\mu| < \|k\|^{-1}_{L^2(|a,b|)\times(|a,b|)}$, then the preceding estimate shows that T is a contraction mapping.

VIII. Matrix Equations

Consider system of linear algebraic equations given by the matrix problem $Ax = b$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{13} \\ a_{21} & a_{22} & \cdots\cdots & a_{23} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & \cdots\cdots & \vdots \\ a_{n1} & a_{2n} & & a_{nn} \end{bmatrix}$$

$$=(x_1, x_2, \ldots\ldots, x_n)^r = b = (b_1, b_{2,\ldots,} b_n)^r$$

We can re-write this system of equations as

$x_1 = (1 - a_{11})x_1 - a_{12}x_2 - \cdots - a_{1n}x_n + b_1$

$x_2 = -a_{21}x_1 + (1 - a_{22})x_2 - \cdots - a_{2n}x_n + b_2$

$\vdots \qquad \vdots$

$x_n = -a_{n1}x_1 - a_{n2}x_2 - \cdots + (1 - a_{nn})x_n + b_n$

For $1 \leq I, j \leq n$, *set* $a_{ij} := -a_{ij} + \delta_{ij}$, where δ_{ij} is the Kronecker delta function. We can write the above system of equations as

$$x_i = \sum_{j=1}^{n} a_{ij}\ x_j + b_i \qquad \forall i=1,\ldots,n$$

We can see that the matrix problem $Ax = b$ is equivalent to the matrix problem $x - Ax + b = x$. Then We define a map $T : R^n \to R^n$, $T(x) := x - Ax + b$

Thus finding solutions of the matrix problem Ax = b is equivalent to finding fixed points of the map T. Observe that, for x $x^i \in R^n$,

$$Tx - Tx' = (x - Ax + b) - (x' - Ax' + b)$$
$$= (x - x') - (Ax - Ax') = (x - x') - A(x - x') = (I - A)(x - x')$$

We claim that $Ax = b$ has a unique solution if

$$\sum_{j=1}^{n} |a|_{ij} \leq k \leq 1. \forall i = 1, \ldots, n$$

We define a metric d on R^n by

$d(y - y') = \sup_{1\leq i\leq n} | (x_i - x_i'|)$. Then

$d(Tx; Tx') = \sup_{1\leq i\leq n} | a_{ij} | x - x'| \leq \sup_{1\leq i\leq n} \sum_{j=1}^{n} | a_{ij} | | x_j - x_j'$

$= \sup_{1\leq i\leq n} (\sum_{j=1}^{n} | a_{ij}|)(\sup_{1\leq i\leq n} | x - x'|)$

$= d(x - x') = \sup_{1\leq i\leq n} \sum | a_{ij}|$

$\leq kd(x - x')$.

Which shows that T is a contraction mapping.

IX. Picard-Lindeöf Theorem

9.1Theorem: Let $A = \{(x, y) \in R^2, a \leq x \leq b, c \leq y \leq d\} \subset R^2$

and let f : A → R be Lipschitz continuous in the second variable. Let $(x_0; y_0) \in A^\circ$. Then the ordinary differential equation

$$\frac{dy}{dx} = f(x, y)$$

has a unique solution $y = g(x)$ satisfying $g(x_0) = y_0$ defined on an interval $[x_0 - \epsilon, x_0 + \epsilon]$, for some $\epsilon > 0$.

Proof:. By the fundamental theorem of calculus, solving the ODE in the statement of the theorem is equivalent to finding a unique solution to the integral equation

$$g(x) = g(x_0) + \int_{x_0}^{x} f(t, g(t))dt$$

Let q > 0 be a constant such that

$| f(x, y_1) \leq f(x, y_2)| \leq q | y_1 - y_2 |$ for all (x, y_1); $(x, y_2) \in A$. Since $A \subset R^2$ is compact and f is continuous, f is bounded some constant M > 0 on A. Choose $\epsilon > 0$ such that $\epsilon < q^{-1}$, and let B

$:= \{(x,y) \in R^2 : x_0 - \epsilon \leq x \leq x_0 + \epsilon,$
$y_0 - M_\epsilon \leq y \leq y_0 + M_\epsilon\}$

Note that $B \subset A$. Let X be the subset of $(C([x_0 - \epsilon; x_0 + \epsilon])$; d), with $d(.,.) = \|. \_. \|_{L^\infty}$, of functions g satisfying $d(g, g(x_0)) \leq M_\epsilon$. It is evident from limit properties that (X, d) is a closed subspace. Set $h := y_0 + \int_{x_0}^{x} f(t, g(t))dt$.

We observe that

d(h, y_0) = sup $_{x\in[}x_{0_{-\epsilon\_;}}x_0-\epsilon|$ $y_{0+}\int_{x_0}^{x} f(t,g(t))dt - y_0|$

$\leq_{\sup}$

$x \in [x_0 - \epsilon_{;x_0} - \epsilon)| \int_{x_0}^{x} | f(t,g(t))| \, dt \leq \sup x \in [x_0 - \epsilon_{;x_0} - \epsilon \int_{x_0}^{x} M dt = M$

Hence, h ∈ X. Define a mapping *T : X →X by Tg := h*. We claim that T is a contraction mapping. Indeed, for g_1, g_2 ∈ X, we have that

d(Tg_1; Tg_2) = sup $_{x\in[}x_{0_{-\epsilon\_;}}$ $x_0 - \epsilon \int_{x_0}^{x} f(t,)$ $g_1(t)$) dt- $y_{0+}\int_{x_0}^{x} f$ (t,g_2(t)dt

≤ sup $_{x\in[}x_{0_{-\epsilon\_;}}$ $x_0 - \epsilon \int_{x_0}^{x} | f(t,)$ g_1(t))−f(t, g_2(t))| dt

≤ sup $_{x\in[}x_{0_{-\epsilon\_;}}$ $x_0 - \epsilon \int_{x_0}^{x} | q|$ g(t))− g_2(t))| dt

≤*d*(g_1, g_2) $sup_{x_2 \in x_0 - \epsilon, x_0 + \epsilon} \int_{x_0}^{x} q dt$

$= \underbrace{q\epsilon}_{k}$ d(g_1, g_2)

where 0 ≤ k ≤1 by our choice of ϵ. We conclude that T is a contraction mapping.

More of applications follows on the next paper.

REFERENCES

[1] Andrzej Granas and James Dugundji, Fixed Point Theory. Springer-Verlag, New York(2003)

[2] 12Banach, S. "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales." Fund. Math. 3(1922).

[3] Hitzler, Pascal; Seda, Anthony K. "A ‘Converse’ of the Banach Contraction Mapping Theorem". Journal of Electrical Engineering 52 (10/s): 3–6, (2001)

[4] Hitzler, Pascal; Seda, Anthony. Mathematical Aspects of Logic Programming Semantics. Chapman and Hall/CRC(2010).

[5] Kirk, William A.; Khamsi, Mohamed A. An Introduction to Metric Spaces and Fixed Point Theory. John Wiley, New York. . (2001).

[6] Larson, Ron, Bruce H. Edwards. Calculus, 9th ed., Brooks Cole Cengage Learning. (2010)

[7] McQuarrie, Donald A.. Mathematical Methods for Scientists and Engineers, University Science Books. (2003)

[8] Palais, R. "A simple proof of the Banach contraction principle." J. fixed point theory appl 2 (2007)

[9] Salas, Saturnino L.; Hille, Einar; Etgen, Garret J. Calculus: One and Several Variables (10th ed.). Wiley.. (2006).

[10] Seda, Anthony K.; Hitzler, Pascal. "Generalized Distance Functions in the Theory of Computation". The Computer Journal 53 (4): 443–464((2010)).

[11] Stewart, James . Calculus: Early Transcendentals, 6th ed., Brooks Cole Cengage Learning. (2008)

[12] Thomas, George B., Maurice D. Weir, Joel Hass, Frank R. Giordano,Calculus, 11th ed., Addison-Wesley. (2008).

[13] Vasile I. Istratescu, Fixed Point Theory, An Introduction, D.Reidel, the Netherlands See chapter 7. (1981).

[14] William A. Kirk and Brailey Sims, Handbook of Metric Fixed Point Theory ,Kluwer Academic, London(2001),.

AUTHORS

First Author – N. H. Mohammed, Math Dept College of Engineering-Sinnar University-Sudan, E.mail: NHMOHAMMED67@ G-MAIL.COM

Second Author – M.A.Bashir, Math Dept El-Neenlain University-Sudan, E-mail:mabashir 09@G-mail.com