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On Fixed Point Theorems and Their Applications (Ⅱ)

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I. Semi linear equations of evolution

1. 1. Strongly continuous semigroups :
Let X be a Banach space. A one parameter family S(t) ($t \geq 0$) of bounded linear operators on X is a strongly continuous semigroup ,
(C_0 is semigroup for short) if
(a) *S(0) = I which I is the identity operator on X.*
(b) $S(t + s) = S(t)S(s)$ for every t, s > 0.
(c) $\lim_{t\to 0} S(t)x = x$ for every $x \in X$ (strong continuity).

As a quite direct application of the uniform boundedness theorem, there exist $\omega \geq 0$ and $M \geq 1$ such that

$\|S(t)\|_{L(X)} \leq M e^{\omega t}$, $\forall\, t \geq 0$.

This in turn entails the continuity of the map $t \mapsto S(t)x$ from $[0,\infty)$ to X, for every fixed $x \in X$ .The linear operator A domain

$D(A)=\{x \in X : \lim_{t\to 0} \frac{S(t)\, x - x}{t}$ exists$\}$

defined by

$Ax = \lim_{t\to 0} \frac{S(t)\, x - x}{t}$, $\forall x \in D(A)$

is the infinitesimal generator of the semigroup S(t).Now we recall some basic facts on A .

Proposition : A is a closed linear operator with dense domain. For every fixed $x \in D(A)$, the map $t \mapsto S(t)x$ belongs to $C^1([0,\infty),D(A))$ and

$\frac{d}{dt} S(t)x = AS(t)x = S(t)Ax.$

We consider the following semilinear Cauchy problem in X:

$$\begin{cases} x'(t) = Ax(t) + f\big(t, x(t)\big), \\ x(0) = x_0 \in X \end{cases} \qquad 0 < t < T \qquad (3)$$

where A is the infinitesimal generator of a strongly continuous semigroup S(t), and $f : [0, T'] \times X \to X$ is continuous and uniformly Lipschitz continuous on X with Lipschitz constant $\lambda \geq 0$.

1.2. Definition :A function x : *[0, T]* × *X* is said to be a classical solution to (3) if it is differentiable on [0, *T) x(t)* $\in D(A)$ for every $t \in [0, T]$, and (3) is satisfied on [0, T].

If x is a classical solution, it is necessarily unique, and it is given by

$$x(t) = S(t)x_0 + \int_0^t S(t-s)f(s,x(s))\, ds. \quad (4)$$

which is easily to proved integrating in *ds* on [0, t] the derivative with respect to s of the differentiable function $S(t-s)x(s)$ and using (3). Notice that above (Riemann) integral is well-defined, since
if $x \in C([0,],X)$ the map $t \mapsto f(t, x(t))$

belong to *C([0, T],X*) as well .Of course, there is no reason why there should exist a classical solution for a certain initial value x_0. However, formula (4) makes sense for every $x \in X$. This motivates the following definition.

1.3. Definition: A function $x : [0, T] \to X$ is a mild solution to (3) if it continuous on [0, T] and fulfills the integral equation (4).

1.4. Theorem: For every $x \in X$ the Cauchy problem (3) has a unique mild solution. Moreover the map $x_0 \mapsto x(t)$ is Lipschitz continuous from X into *C([0, T],X).*

Proof :
For a given $x_0 \in X$ define the map $F : C([0, T],X) \to C([0, T],X)$ by

$$F(x)(t) = S(t)x_0 + \int_0^t S(t-s)f(s,x(s))\, ds.$$

Then we have

$\| F(x)(t) - F(y)(t) \|_X \leq \lambda M t \, \| x - y \|_{C([0,T],X)}$

where $M = \sup_{t\in[0,T]} \| S(t) \|_{L(X)}$. By an inductive argument, analogous to the one used in the proof of Theorem 3.4, (in paper one) we get

$$\|F^n(x)(t) - F^n(y)(t)\|_X \leq \frac{(\lambda M T)^n}{n!} \|x - y\|_{C([0,T],X)}.$$

Hence for $n \in N$ big enough F^n is a contraction, so by Corollary 3.3 (in paper one) F has a unique fixed point in C([0, T], X) which is clearly the desired mild solution to the Cauchy problem (3).

To complete the proof, let y be the unique mild solution corresponding to the initial value y_0. Then

$$\|x(t) - y(t)\|_X \leq M \|x_0 - y_0\|_X + \lambda M \int_0^t \|x(s) - y(s)\|_X\, ds.$$

By the Gronwall Lemma , we get at once
$\|x(t) - y(t)\|_X (M e^{\lambda M T} \|x_0 - y_0\|_X$, $\forall\, t \in [0, T]$

which entails the Lipschitz continuity of the map $x_0 \mapsto x(t)$.

1.5. The abstract elliptic problem:

Suppose that X, V are Banach spaces with compact and dense embeddings $V \to X \to V^*$. Assume we are given a bounded linear operator

$A : V \to V^*$ and a nonlinear continuous map $B : X \to V^*$ carries bounded sets into bounded sets, such that

$\langle Au, u\rangle \geq \varepsilon \|u\|_v^2$, $\forall u \in V$ (5)

and

$\langle B(u), u\rangle \geq -c(1 + \|u\|_v^{\alpha})$, $\forall u \in X$ (6)

for some $\epsilon > 0$, $c \geq 0$ and $\alpha \in [0, 2)$, where $\langle \cdot, \cdot\rangle$ denotes the duality pairing between V^* and V .

Problem: Given $g \in V^*$, find a solution $u \in V$ to the abstract equation

$Au + B(u) = g$. (7)

For $v \in X$, let w is the solution to the equation

$Aw = g - B(v)$.

From (5) A is injective onto V , and by the open mapping theorem, A^{-1} is a bounded linear operator from V onto V .

Therefore

$w = A^{-1}(f - B(v)) \in V$.

Define the map $f : X \to X$ as $f(v) = A^{-1}(g - B(v))$. Notice that f is continuous and compact. Suppose then that, for some λ, there is u_λ such that $u_\lambda = \lambda\ f(u_\lambda)$. This means that u solves the equation

$Au_\lambda + \lambda B(u_\lambda) = \lambda g$.

Taking the duality pairing of the above equation and u, and exploiting (5)-(6),we get

$\varepsilon \|u_\lambda\|_v^2 \leq \lambda c(1 + \|u_\lambda\|_v^{\alpha} + \lambda\|g\|_{V^*}\|u_\lambda\|_V$.

Recalling now that $\lambda \in [0, 1]$, and using the Young inequality

$ab \leq_\nu a^p + K(\nu, p) b^q$ $(a, b \geq 0, \nu > 0)$

where $K(\nu, p) = (\nu p)^{-q/p} q^{-1}$ $(1 < p, q < \infty, 1/p + 1/q = 1)$, we find the a priori estimate

$\|u_\lambda\|_v^2 \leq \frac{2}{\varepsilon}$ ($c + K(\varepsilon/4, 2/\alpha)\, c^{2/(2-\alpha)} + \frac{1}{\varepsilon} \|g\|_{V^*}^2$) .

which is clearly a solution to (7). Notice that the solution might not be unique.

1.6. Example : Let $\Omega \subset R^2$ be a bounded domain with smooth boundary $\partial\Omega$. Find a weak solution to the nonlinear elliptic problem

$$\begin{cases} -\Delta u + u^5 = g \\ u_{|\partial\Omega} = 0 \end{cases}$$.

In this case $V = H_1^{0}(\Omega)$, $X = L^6(\Omega)$,

and $g \in H^1{}_0(\Omega)^* = H^{-1}(\Omega)$, for the definitions and the properties of the Sobolev spaces $H^1{}_0$ and H^{-1},in particular, we recall that the embedding $H^1{}_0(\Omega) \hookrightarrow L^6(\Omega)$ is compact when $\Omega \subset R^2$. Then $A = -\Delta$ and $B(u) = u^5$ are easily seen to fulfill the required hypotheses. .

(1.7)The invariant subspace problem:

The invariant subspace problem is probably the problem of operator theory. The question, that attracted the attention of great deal of mathematicians is quite simple to state . Given a Banach space X and an operator $T \in L(X)$, find a closed nontrivial subspace M of X (i.e., $M \neq X$ and $M \neq \{0\}$) for which $TM \subset M$. Such M is an invariant subspace for T. It is known that not all continuous linear operators on Banach spaces have invariant subspaces. The question is still open for Hilbert spaces.

The most general, and at the same time most spectacular, result on the subject, is the Lomonosov theorem, that provides the existence of hyperinvariant subspaces for a vast class of operators. The proof is relatively simple, and the role played by the Schauder-Tychonoff theorem is essential. In order to state the result, we want first a definition.

1.8 Definition: suppose that X is a Banach space. An invariant subspace M for $T \in L(X)$ is said to be hyperinvariant if it is invariant for all operators commuting with T (that is, for all $T' \in L(X)$ such that

$TT' = T'T$).

Remark: If $T \in L(H)$ is non scalar, i.e., is not a multiple of the identity, and it has an eigenvalue λ , then the eigenspace M corresponding to λ is hyperinvariant for T. if $x \in M$ and T^0 commutes with T, we have that

$\lambda T'x = T'Tx = TT'x$.Therefore $T'x \in M$.

1.9. Theorem [Lomonosov]:

Suppose that X is a Banach space. Let $T \in L(X)$ be a nonscalar operator commuting with a nonzero compact operator $S \in L(X)$. Then T has a hyperinvariant subspace.

We anticipate an observation that will be used in the proof.

Remark: Suppose that $S \in L(X)$ is a compact operator . If $\lambda \neq 0$ is an eigenvalue of S, then the eigenspace

$F := \{x \in X : Sx = \lambda x\}$

relatively to λ has finite dimension. Indeed, the restriction of S on F is a (nonzero) multiple of the identity on F, and the identity is compact if and only if the space is finite-dimensional.

Proof:

We proceed by contradiction. Let A be the algebra of operators commuting with T. It is immediate to see that if T has no hyperinvariant subspaces, then $\overline{Ax}$ = X for every x ∈ X, x ≠0.
Without loss of generality, let $\| S \|_{L(X) \leq 1}$. Choose x_0 ∈ X such that $\| Sx_0 \| > 1$ (which implies $\| x_0 \| > 1$) and set $B = \bar{B}_X(x_0, 1)$. For x ∈ $\overline{SB}$ and notice that x cannot be the zero Vector , there is T' ∈ A such that
$\| T'x - x_0 \| < 1$.Hence every x ∈ $\overline{SB}$ has an open neighborhood Vx such that T'Vx⊂ B for some T' ∈ A. Exploiting the compactness of $\overline{SB}$, we find a finite *cover* $V_1, \dots, V_n$ and $T_1', \dots, T_n'$ ∈ A such that $T'_j V_j \subset B$, ∀ j = *1, . . . , n.*
Let $\varphi_1, \dots, \varphi_n \in C(\overline{SB})$ be a partition of the unity for $\overline{SB}$ subordinate to the open cover {V_j}, and define, for x ∈ B,

$$f(x) = \sum_{j=1}^{n} \varphi j(Sx) T_j' j S x.$$

Then f is a continuous function from B into B. Since T_j' S is a compact map for every j, it is easily seen that f(B) is relatively compact. Hence, that by the existence of $\bar{x}$ ∈ B such that $f(\bar{x}) = \bar{x}$. Defining the operator $\bar{T}$ ∈ A as

$$\bar{T} = \sum_{j=1}^{n} \varphi j(S\bar{x}) T_j'$$

we get the relation

$$\bar{T}_S \bar{x} = \bar{x}.$$

But $\bar{T}$S is a compact operator, hence the eigenspace F of $\bar{T}$S relative to the eigenvalue 1 is finite-dimensional. Since $\bar{T}$S commutes with T, we conclude that F is invariant for T, which means that T has an eigenvalue, and thus a hyperinvariant subspace, contrary to our assumption.

1.10. Measure preserving maps on compact Hausdorff spaces :

Suppose that X is a compact Hausdorff space, and let P(X) be a set of all Borel probability measures on X. By means of the Riesz representation theorem, the dual space of C(X) can be identified with the space M(X) of complex regular Borel measures on X. Then the norm
‖ μ ‖ of an element μ ∈ M(X) is given by the total variation of μ. It is straightforward to check that P(X) is convex and closed in the weak* topology. Moreover, P(X) is weak* compact. Indeed, it is a weak* closed subset of the unit ball of *M(X)*, which is weak* compact by the Banach-Alaoglu theorem.

111. Definition :

Suppose that μ ∈ P(X). A μ-measurable map f : X → X is measure preserving with respect to μ if μ (B) = μ (f^{-1}(B)) for every Borel set
B ⊂ X. Such μ is an invariant measure for f.

Notice that, if f is a μ-measurable map, the measure $\bar{f}\mu$ defined by

$$\bar{f}\mu(B) = \mu(f^{-1}(B)), \ \forall \text{ Borel set } B$$

belongs to *P(X)*. In particular, if *f* is continuous (and therefore measurable with respect to every μ ∈ P(X)), we have a map $\bar{f}$: *P(X) →P(X)* defined by μ ↦ $\bar{f}$μ. In addition, applying the monotone convergence theorem to an increasing sequence of simple functions, it is easy to see that

$$\int_x g \, d(\bar{f}\mu) = \int_x g \circ f \, d\mu, \ \forall g \in C(X).$$

Lemma: Assume *f : X → X* be continuous. Then the map $\bar{f}$: *P(X) → P(X)* is continuous in the weak* topology.
proof Let $\{\mu_s\}_{s\in I} \subset P(X)$ be a net converging to some μ ∈ P(X). Then, for every g ∈ C(X),

$$\lim_{s\in I} \int_x g \, d(\bar{f}\mu_s) = \lim_{s\in I} \int_x g \circ f \, d\mu_s = \int_x g \circ f \, d\mu = \int_x g \, d(\bar{f}\mu)$$

which entails the claimed continuity.
The finding to elements of P(X) that are invariant measures for f. This is the same as finding a fixed point for the map $\bar{f}$.

II. Invariant Means on Semigroups

Let S is a semigroup, that is, a set endowed with an associative binary operation, and consider the (real) Banach space of all real-valued bounded functions on S, namely, ℓ^∞(S) ={f : S → R : ‖f ‖ := $\sup_{s\in S}$|f(s)| < ∞}.An element f ∈ ℓ^∞(S) is positive if f(s) ≥ 0 for every s ∈ S. A linear functional ∧ : ℓ^∞ (S) → R is positive if ∧f ≥ 0 for every positive element f ∈ ℓ^∞ (S). We agree to denote a constant function on S by the value of the constant. We now recall a result that actually holds for more general situations.

Lemma :Let ∧ ∈ ℓ^∞ (S)*, with ‖ ∧‖ =∧1 = 1. Then ∧ is positive.

Proof: Assume not. Then there is f ∈ ℓ^∞ (S), f ≥ 0, such that
∧ f = β < 0. For ε > 0 small, we have
‖1 − ε f‖ = $\sup_{s\in S}$|1 − ε f(s)|≤ 1.
Hence
1 < 1 − ε β |1 − ε β| = |∧| (1 − ε f)| ≤‖ 1 − ε f ‖≤ 1
Leading to a contradiction.

If $t \in S$, then we can define the left t-translation operator $L_t : \ell^\infty(S) \to \ell^\infty(S)$ to be
$(L_t f)(s) = f(ts), \forall s \in S$.

In an analogous manner, we can define the right t-translation operator R_t.

2 .1. Definition: A left invariant mean on S is a positive linear functional $\wedge$ on $\ell^\infty(S)$ satisfying the following conditions:
(a) $\wedge 1 = 1$;
(b) $\wedge(L_s f) = \wedge f$ for every $s \in S$ and every $f \in \ell^\infty(S)$.

When such a functional exists, S is (left) amenable.

Clearly, we can give the above definition replacing left with right or two-sided. The distinction is relevant if S is not abelian.

2. 2. Example [Banach]: suppose that S = N. Then $\ell^\infty(N) = \ell^\infty$. An invariant mean in this case is called a Banach generalized limit. The reason is that if $\wedge$ is an invariant mean on N and
$x = \{x_n\}_{n\in N} \in \ell^\infty$ is such that $\lim_{n\to\infty} x_n = \alpha \in R$, then $x \wedge x = \alpha$.
Indeed, for any $\varepsilon > 0$, we can choose n_0 such that
$\alpha - \varepsilon \le x_n \le \alpha + \varepsilon$ for every $n \ge n_0$. Hence, if we define
$y = \{y_n\}_{n\in N} \in \ell^\infty$ by
$y_n = x_n + n_0$,
we have
$\wedge x = \wedge y$,
and
$\alpha - \varepsilon = \wedge(\alpha - \varepsilon) \le \wedge y \le \wedge(\alpha + \varepsilon) \le \alpha + \varepsilon$
which yields the equality $\wedge x = \alpha$. To prove the existence of an invariant mean, one has to consider the subspace $\mathcal{M}$ of ℓ^∞ given by

$$\mathcal{M} = \{x = \{x_n\}_{n\in N} \in \ell^\infty : \lim_{n\to\infty} \frac{x_1 + \cdots + x_n}{n+1} = \alpha_x \in R\}$$

and define the linear functional $\wedge_0$ on M as $\wedge_0 x = \alpha_x$. Setting for every
$x \in \ell^\infty$

$$p(x) = \limsup_{n\to 1} \frac{x_1 + \cdots + x_n}{n+1}$$

it is then possible, by means of the Hahn-Banach theorem, to extend $\wedge_0$ to a functional $\wedge$ defined on the whole space, in such a way that
$-p(-x) \le x \le p(x)$ for every $x \in \ell^\infty$. In particular, $\wedge$ is continuous.

A remarkable consequence of this fact is that not every continuous linear functional on ℓ^∞ can be given the representation
$\{x_n\}_{n\in N} \mapsto \sum_{n=0}^{\infty} c_n x_n$
for some numerical sequence c_n. Indeed, for every $k \in N$, taking
$e_k = \{\delta_{nk}\}_{n\in N}$,
we have $\wedge e_k = 0$. Hence if $\wedge$ has the above representation, all the c_n must be zero, i.e. $\wedge$ is the null functional, contrary to the fact that $\wedge 1 = 1$.

2.3. Theorem: **[Day]**: Suppose that S is an abelian semigroup. Then S is amenable.

proof : Denote $K = \{\wedge \in \ell^\infty(S)^* : \|\wedge\| = \wedge 1 = 1\}$.
In particular, if $\wedge \in K$ then $\wedge$ is positive. K is convex, and from the Banach- Alaoglu theorem is compact in the weak* topology of $\ell^\infty(S)^*$. We define the family of linear operators
$T_s : \ell^\infty(S)^* \to \ell^\infty(S)^*$, $s \in S$, as $(T_s\wedge)(f) = \wedge(L_s f)$, $\forall f \in \ell^\infty(S)$.

First we show that T_s is continuous in the weak* topology for every $s \in S$. Of course, it is enough to show the continuity at zero. Thus let V be a neighborhood of zero of the local base for the weak* topology, that is,
$V = \{\wedge \in \ell^\infty(S)^* : |\wedge f_j| < \varepsilon_j, j = 1, \dots, n\}$
for some $\varepsilon_1, \dots, \varepsilon_n > 0$ and $f_1, \dots, f_n \in \ell^\infty(S)$. Then
$T_s^{-1}(V) = \{\wedge \in \ell^\infty(S)^* : |(T_s\wedge)(f_j)| < \varepsilon_j, j = 1, \dots, n\}$
$= \{\wedge \in \ell^\infty(S)^* : |\wedge(L_s f_j)| < \varepsilon_j, j = 1, \dots, n\}$
is an open neighborhood of zero .The second step is to prove that $T_s K \subset K$. Indeed, $(T_s\wedge)(1) = \wedge(L_s 1) = \wedge 1 = 1$
And
$\|T_s \wedge\| = \sup_{\|f\|\le 1} |(T_s\wedge)(f)| = \sup_{\|f\|\le 1} |\wedge(L_s f)| \le \sup_{\|f\|\le 1} |\wedge f|$

Since $\|L_s f\| \le \|f\|$ .Finally, for every $s, t \in S$,
$T_s T_t \wedge = T_s(\wedge_o L_t) = \wedge_o L_{t\,o} L_s = \wedge_o L_{st} = \wedge_o L_{ts} = T_t T_s \wedge$.
Hence there is $\wedge \in K$ such that $T_s \wedge = \wedge$ for every $s \in S$, which means that $\wedge(L_s f) = \wedge f$ for every $s \in S$ and every $f \in \ell^\infty(S)$.

III. Ordinary differential equations in Banach spaces

3.1. The Riemann integral:

Suppose that X is a Banach space, $I = [\alpha, \beta] \subset R$. The notion of Riemann integral and the related properties can be extended with no differences from the case of real-valued functions to X-valued functions on I.

In particular, if $f \in C(I,X)$ then f is Riemann integrable on I,
$\left\| \int_\alpha^\beta f(t)\, .dt \right\|_x \le \int_\alpha^\beta \|f(t)\|_x\, dt$ and
$\frac{d}{dt} \int_\alpha^t f(y)\,dy = f(t)\ \forall t \in I$.

Recall that a function $h : I \to X$ is differentiable at $t_0 \in I$ if the $\lim_{t\to t_0} \frac{h(t) - h(t_0)}{t - t_0}$ exists in X. This limit is the

derivative of h at t_0, and is denoted by $\dot{h}'(t_0)$ or $\frac{dh(t_0)}{dt_0}$. If t∈ (α, β) we recover the definition of Fr´echet differentiability. It is easy to see that if
$h'(t) = 0$ for all t ∈[α, β], then h(t) is constant on [α, β]. Indeed, for every $\wedge \in X^*$, we have $(\wedge \circ h(t)' = \wedge' h(t) = 0$, that implies $\wedge(h(t) - h(\alpha)) = 0$, and from the Hahn-Banach theorem there is $\wedge \in X^*$ such that $\wedge(h(t) - h(\alpha)) = \|h(t) - h(\alpha)\|$.

3.2. The Cauchy problem :

Suppose that X is a Banach space, U ⊂ R × X, U open, $u_0 = (t_0, x_0) \in U$, f : U → X continuous. The problem is to find a closed interval I, with t_0 belonging to the interior of I and a differentiable function x : I →X such that

$$\begin{cases} x' = f(t, x(t)) & t \in I \\ x(t_0) = x_0 \end{cases} \quad (1)$$

It is apparent that such x is automatically of class C^1 on I. Also, it is readily seen that (1) is equivalent to the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(y, x(y))\,dt \quad t \in I \qquad (2)$$

Namely, x is a solution to (1) if and only if it is a solution to (2).

3.3. Theorem :[Local solution]

Assume the following hypotheses:

(a) *f* is continuous;

(b) The inequality $\|f(t,x_1) - f(t,x_2)\|_x \le k(t\|x_1 - x_2\|_x \forall (t,x_1),(t,x_2) \in U$

holds for some k(t) ∈ [0,∞];

(c) $k \in L^1((t_0 - a, t_0 + a))$ for some a > 0;

(d) There exist m ≥ 0 and $\bar{B}_{R\times X}(u_0, s) \subset U$ such that

$\|f(t,x)\|_x \le m$, ∀ (t, x) ∈ $\bar{B}_{R\times X}(u_0, s)$.Then there is $\tau_0 > 0$ such that, for any $\tau < \tau_0$, there exists a unique solution $x \in (C^1(I_r, X)$ to Cauchy problem (3.2) which $I_r = [t_0 - r, t_0 + r]$,

Remark :

Notice first that from (b), since U is open, k is defined in a neighborhood of zero. If k is constant then (c)-(d) are automatically satisfied. Indeed, for (x, t) ∈ $\bar{B}_{R\times X}(u_0, s)$, we have $\|f(t), x\|_x \le k_s + \max_{t - t_0 \le s} \|f(t), x_0\|_x$

Also, (d) is always true if X is finite-dimensional, for closed balls are compact. In both cases, setting $s_0 = \sup\{\sigma > 0 : \bar{B}_{R\times X}(u_0, \sigma) \subset U$ we can choose any s < s_0.

Proof:

Suppose that $r = \min\{a, s\}$, and set $\tau_0 = \min\{r, \frac{r}{m}\}$. Select then $\tau < \tau_0$, and consider the complete metric space $Z = \bar{B}_{C(I_r,X)}(x_0, r)$ with the metric induced by the norm of $C(I_r, X)$ (here x_0 is the constant function equal to x_0).

Since T < r, if z ∈ Z then $(t, z(t)) \in \bar{B}_{R\times X}(u_0, r) \subset U$ for all $t \in I_r$ .Hence, for z ∈ Z, define

$F(z)(t) = x_0 + \int_{t_0}^{t} f(y, z(y)dy)$ $t \in I_r$. we Observe that $\sup_{t \in I_r} \|F(z)(t) - x_0\|_x \le \sup_{t \in I_r} |\int_{t_0}^{t} \|f(y.z(y))\|_x\,dy| \le m_r \le r$

We conclude that F maps Z into Z. The last step is to show that F_n is a contraction on Z for some n ∈N. By induction on n we show that, for every $t \in I_r$,

$$\|F^m(y), z(t) - F^n(z_t)(t)\|_x \le \frac{1}{n!} \int_{t_0}^{t} k(y)dy|^n \|z_1 - z_2\|_{x(I_{r_x})} \quad (3)$$

For n = 1 it holds easily. So assume it is true for n − 1, n ≥ 2. Then, taking $t > t_0$ and the argument for $t < t_0$ is analogous,

$$\|F^n(z_1)(t) - F^n(z_2)(t)\|_x$$

$$= \|FF^{n-1}(z_1)(t) - FF^{n-1}(z_2)(t)\|_x$$

$$\le \int_{t_0}^{t} \|f(y)F^{n-1}(z_1)(y) - f(y)F^{n-1}(z_2)(y)\|_x dy$$

$$\le \int_{t_0}^{t} k(y)\,\|F^{n-1}(z_1)(y)\|_x\,dy$$

$$\le \frac{1}{(n-1)!}[\int_{t_0}^{t} k(y)(\int_{t_0}^{t} k(w)dw)_{n-1}dy]\,\|z_{1-z_2}\|_{c(I_r,x)}$$

$$= \frac{1}{n!}(\int_{t_0}^{t} k(y)dy)_n \|z_1 - z_2\|_{c(I_r,x)}$$

There fore from(3) we get

$$\|F^m(z_1) - F^n(z_2)\|_{c(I_r x)} \leq \frac{1}{nl} \|k\|_{l^1}^n \|z_1 - z_2\|_{c(I_{r,x})}.$$

Then we show that when n is big enough F^n is a contraction. By means of Corollary 3.3 in paper(1),we conclude that F admits a unique fixed point, it is clearly the unique solution to the integral equation (2) and hence to (1).

3.4. Proposition :[**Continuous dependence]** The solution to the Cauchy problem 3.2 depends with continuity on the initial data.

Proof:

By Assuming that $x_j \in C(I_r, X)$ are two solutions to (1) with initial data x_{0j} (j = 1, 2). Setting $x = x_1 - x_2$ and $x_0 = x_{01} - x_{02}$, from (b) then we get

$$\|x(t)\|_x \leq \|x_0\|_x + \left| \int_{t_0}^t k(y)\|x(y)\|_x \, dy \right| \quad \forall\, t \in I_r \quad (4)$$

The positive function

$$\psi(t) = \|x_*\|_x \exp\left[\left|\int_{t_0}^t k(y)dy\right|\right]$$

satisfies the equation

$$\psi(t) = \|x_0\|_x + \left|\int_{t_0}^t k(y)\psi(y)dy\right| \quad \forall t \in I_r \quad (5)$$

By the comparing of (4) with (5), we conclude that

$$\|x(t)\|_x \leq \|x_0\|_x \exp\left[\left|\int_{t_0}^t k(y)dy\right|\right] \ \forall\, t \in I_r \ . (6)$$

Indeed, defining $\psi = \|x\|_x - \varphi$, addition of (4) and (5) entails

$$\psi(t) \leq \operatorname{sgn}(t - t_0)\int_{t_0}^t k(y)\psi(y)dy, \ \forall t \in I_r .$$

we show that the above inequality implies $\psi \leq 0$ on $[t_0, t_0 + \rho]$ for any $\rho < r$, an thus on $[t_0, t_0 + \tau)$ and the argument for $(t_0 - \tau]$ is the same. Choose n big enough such that

$$\int_{t_0 + \frac{jp}{n}}^{t_0 + \frac{(j+1)p}{n}} k(y)dy < 1 \ \forall\ j = 0,1,\ldots,n-1$$

To finish the proof, we use an inductive argument. Suppose we proved that $\psi \leq 0$ on $[t_0, t_0 + j\rho/n]$ for some $j \leq n - 1$, and let t^* be such that

$$\psi(t^*) = \max\{\psi(t) : t \in [t_0 + \frac{j\rho}{n}, t_0 + (j+1)p/n]\}.$$

Then

$$\psi(t^*) \int_{t_0}^t k(y)\psi(y)dy \leq \psi(t^*)\int_{t_0}^{t_0} k(y)dy$$

So if $\psi(t^*)$ is strictly positive, we may cancel (t^*) in the above inequality, getting that the integral exceeds 1. Therefore $\psi \leq 0$ on $[t_0, t_0 + (j + 1)\rho/n]$.

Remark : The implication (4)$\Longrightarrow$(6) is known as the Gronwall lemma, which can also be proved via differential techniques.

3.5. Theorem :[**Global solution**]

Suppose that $U = (\alpha, \beta) \times X$. Assume (a) and (b) of Theorem 3.3, and replace (c) with

$(\grave{c})$ $k \in L^1(\alpha, \beta)$). Then there exists a unique solution $x \in C^1(I,X)$ to the Cauchy problem 3.2, for every $I \subset (\alpha, \beta)$.

proof :Proceed like in the proof of Theorem 3.3, taking now Z = C(I,X).When f is merely continuous and fulfills a compactness property, it is possible to provide an existence result, exploiting the Schauder- Tychonoff fixed point theorem.

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