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Basic Integration on Smooth Manifolds and Applications maps With Stokes Theorem

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Abstract- In this paper of Riemannian geometry to pervious of differentiable manifolds $(\partial M)_p$ which are used in an essential way in basic concepts of Riemannian geometry we study the defections, examples of the problem of differentially projection mapping parameterization $\varphi(U_i)^{-1}$ system by strutting rank k on surfaces $n-k$ dimensional is sub manifolds space of R^n, we prove that in depends only on the doubling charts mapping manifolds, sub manifolds its quantitative extrinsic dimension.

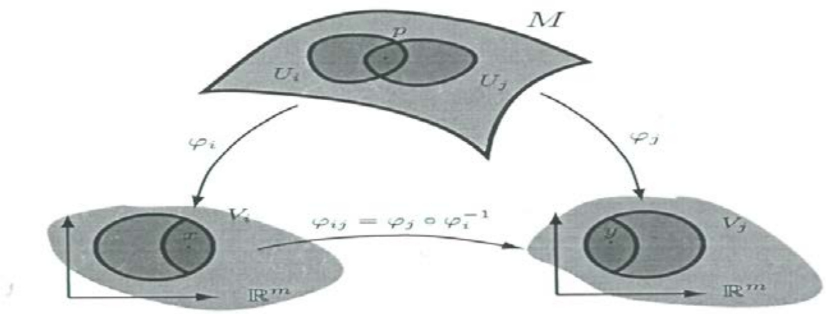
Index Terms- the Euclidean in E^n in that every point has a neighbored, called a chart homeomorphism to an open subset of R^n such as differentiability to basic definitions and properties as a smooth manifold[1] , , transitions between different choices of coordinates are called transitions maps $\varphi_{ij} = \varphi_j \circ \varphi_i$ which are homeomorphisms [2] , A sub manifolds of others of R^n for instance S^2 is sub manifolds of R^3 it can be obtained as the image of map into R^3 is sub Riemannian manifolds[3]

I. Introduction

THis a manifolds is a generalization of curves and surfaces to higher dimension , it is Euclidean in E^n in that every point has a neighbored, called a chart homeomorphism to an open subset of R^n , the coordinates on a chart allow one to carry out computations as though in a Euclidean space , so that many concepts from R^n , such as differentiability, point derivations , tangents , cotangents spaces , and differential forms carry over to a manifold. In this paper we given the basic definitions and properties of a smooth manifold and smooth maps between manifolds , initially the only way we have to verify that a space , we describe a set of sufficient conditions under which a quotient topological space becomes a manifold is exhibit a collection of C^∞ compatible charts covering the space becomes a manifold , giving us a second way to construct manifolds , a topological manifolds C^∞ analytic manifolds , stating with topological manifolds , which are Hausdorff second countable is locally Euclidean space We introduce the concept of maximal C^∞ atlas , which makes a topological manifold into a smooth manifold , a topological manifold is a Hausdorff , second countable is local Euclidean of dimension n .If every point p in M has a neighborhood U such that there is a homeomorphism φ from U onto a open subset of R^n . We call the pair a coordinate map or coordinate system on U . We said chart (U,φ) is centered at $p \in U$, $\varphi(p)=0$, and we define the smooth maps $f: M \to N$ where M,N are differential manifolds we will say that f is smooth if there are atlases (U_α, h_α) on M and (V_β, g_β) on N .

II. DIFFERENTIABLE MANIFOLDS CHARTS

In this section, the basically an m-dimensional topological manifold is a topological space M which is locally homeomorphic to R^m , definition is a topological space M is called an m-dimensional (topological manifold) if the following conditions hold. (i) M is a hausdorff space. (ii) for any $p \in M$ there exists a neighborhood U of P which is homeomorphic to an open subset $V \subset R^m$. (iii) M has a countable basis of open sets , Figurer (1) coordinate charts (U,φ) Axiom (ii) is equivalent to saying that $p \in M$ has a open neighborhood $U \in P$ homeomorphic to open disc D^m in R^m , axiom (iii) says that M can covered by countable many of such neighborhoods , the coordinate chart (U,φ) where U are coordinate neighborhoods or charts and φ are coordinate .
A homeomorphisms , transitions between different choices of coordinates are called transitions maps $\varphi_{ij} = \varphi_j \circ \varphi_i$, which are again homeomorphisms by definition , we usually write $p = \varphi^{-1}(x), \varphi: U \to V \subset R^n$ as coordinates for U , see Figure (1) , and $p = \varphi^{-1}(x), \varphi^{-1}: V \to U \subset M$ as coordinates for U , the coordinate charts (U,φ) are coordinate neighborhoods, or charts , and φ are coordinate homeomorphisms , transitions between different choices of coordinates are called transitions maps $\varphi_{ij} = \varphi_j \circ \varphi_i$ which are again homeomorphisms by definition , we usually $x = \varphi(p), \varphi: U \to V \subset R^n$ as a parameterization U .A collection $A = \{(\varphi_i, U_i\}_{i \in I}$ of coordinate chart with $M = \cup_i U_i$ is called atlas for M .



Figurer (1) : $\varphi_{i\,j}$ $:\varphi_j \circ \varphi_i^{-1}$ the transition maps

The transition maps $\varphi_{i\,j}$ Figurer (1) a topological space M is called (hausdorff) if for any pair $p, q \in M$, there exist open neighborhoods $p \in U$ and $q \in U'$ such that $U \cap U' \neq \phi$ for a topological space M with topology $\tau \in U$ can be written as union of sets in β , a basis is called a countable basis β is a countable set .

A topological space M is called an m-dimensional topological manifold with boundary $\partial M \subset M$ if the following conditions (i) M is hausdorff space (ii) for any point $p \in M$ there exists a neighborhood U of p which is homeomorphism to an open subset $V \subset H^m$ (iii) M has a countable basis of open sets , Figure (3) can be rephrased as follows any point $p \in U$ is contained in neighborhood U to $D^m \cap H^m$ the set M is a locally homeomorphism to R^m or H^m the boundary $\partial M \subset M$ is subset of M which consists of points p .

2.1 Definition

Let X be a set a topology U for X is collection of X satisfying (i) ϕ and X are in U (ii) the intersection of two members of U is in U (iii) the union of any number of members U is in U . The set X with U is called a topological space the members $U \in u$ are called the open sets . let X be a topological space a subset $N \subseteq X$ with $x \in N$ is called a neighborhood of x if there is an open set U with $x \in U \subseteq N$, for example if X a metric space then the closed ball $D_\varepsilon(x)$ and the open ball $D_\varepsilon(x)$ are neighborhoods of x a subset C is said to closed if $X \setminus C$ is open

2.3 Definition

A function $f : X \rightarrow Y$ between two topological spaces is said to be continuous if for every open set U of Y the pre-image $f^{-1}(U)$ is open in X .

2.4 Definition

Let X and Y be topological spaces we say that X and Y are homeomorphic if there exist continuous function such that $f \circ g = id_y$ and $g \circ f = id_X$ we write $X \cong Y$ and say that f and g are homeomorphisms between X and Y, by the definition a function $f : X \rightarrow Y$ is a homeomorphisms if and only if .(i) f is a bijective .(ii) f is continuous (iii) f^{-1} is also continuous.

2.5 Differentiable manifolds

A differentiable manifolds is necessary for extending the methods of differential calculus to spaces more general R^n a subset $S \subset R^3$ is regular surface if for every point $p \in S$ the a neighborhood V of P is R^3 and mapping $x : u \subset R^2 \rightarrow V \cap S$ open set $U \subset R^2$ such that. (i) x is differentiable homomorphism. (ii) the differentiable $(dx)_q : R^2 \rightarrow R^3$, the mapping x is called a parametnzation of S at P the important consequence of differentiable of regular surface is the fact that the transition also example below if $x_\alpha : U_\alpha \rightarrow S^1$ and $x_\beta : U_\beta \rightarrow S^1$ are $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = w \neq \phi$, the mappings $x_\beta^{-1} \circ x_\alpha : x^{-1}(w) \rightarrow R^2$ and .

(2.1) $$x_{\alpha}^{-1} \circ x_{\beta} = x_{\beta}^{-1}(w) \to R$$

Are differentiable . A differentiable structure on a set M induces a natural topology on M it suffices to $A \subset M$ to be an open set in M if and only if $x_{\alpha}^{-1}(A \cap x_{\alpha}(U_{\alpha}))$ is an open set in R^{n} for all α it is easy to verify that M and the empty set are open sets that a union of open sets is again set and that the finite intersection of open sets remains an open set. Manifold is necessary for the methods of differential calculus to spaces more general than de R^{n} , a differential structure on a manifolds M induces a differential structure on every open subset of M , in particular writing the entries of an $n \times k$ matrix in succession identifies the set of all matrices with $R^{n,k}$, an $n \times k$ matrix of rank k can be viewed as a k-frame that is set of k linearly independent vectors in R^{n} , $V_{n,k} K \leq n$ is called the steels manifold ,the general linear group $GL(n)$ by the foregoing $V_{n,k}$ is differential structure on the group n of orthogonal matrices, we define the smooth maps function $f : M \to N$ where M, N are differential manifolds we will say that f is smooth if there are atlases (U_{α}, h_{α}) on M , (V_{B}, g_{B}) on N , such that the maps $g_{B} f h_{\alpha}^{-1}$ are smooth wherever they are defined f is a homeomorphism if is smooth and a smooth inverse. A differentiable structures is topological is a manifold it an open covering U_{α} where each set U_{α} is homeoomorphic, via some homeomorphism h_{α} to an open subset of Euclidean space R^{n} , let M be a topological space , a chart in M consists of an open subset $U \subset M$ and a homeomorphism h of U onto an open subset of R^{m} , a C^{r} atlas on M is a collection (U_{α}, h_{α}) of charts such that the U_{α} cover M and h_{B}, h_{α}^{-1} the differentiable

2.6 Definition (Differentiable injective manifold)

A differentiable manifold of dimension N is a set M and a family of injective mapping $x_{\alpha} \subset R^{n} \to M$ of open sets $u_{\alpha} \in R^{n}$ into M such that.

I. $u_{\alpha} x_{\alpha}(u_{\alpha}) = M$

II. for any α, β with $x_{\alpha}(u_{\alpha}) \cap x_{\beta}(u_{\beta})$

III. the family (u_{α}, x_{α}) is maximal relative to conditions (I),(II) the pair (u_{α}, x_{α}) or the mapping x_{α} with $p \in x_{\alpha}(u_{\alpha})$ is called a parameterization , or system of coordinates of M , $u_{\alpha} x_{\alpha}(u_{\alpha}) = M$ the coordinate charts (U, φ) where U are coordinate neighborhoods or charts , and φ are coordinate homeomorphisms transitions are between different choices of coordinates are called transitions maps.

(2.4) $$\varphi_{i,j} : (\varphi_{j} \circ \varphi_{i}^{-1})$$

Which are anise homeomorphisms by definition , we usually write $x = \varphi(p), \varphi : U \to V \subset R^{n}$ collection U and $p = \varphi^{-1}(x), \varphi^{-1} : V \to U \subset M$ for coordinate charts with is $M = \cup U_{i}$ called an atlas for M of topological manifolds.

A topological manifold M for which the transition maps $\varphi_{i,j} = (\varphi_{j} \circ \varphi_{i})$ for all pairs φ_{i}, φ_{j} in the atlas are homeomorphisms is called a differentiable , or smooth manifold , the transition maps are mapping between open subset of R^{m} , homeomorphisms between open subsets of R^{m} are C^{∞} maps whose inverses are also C^{∞} maps , for two charts U_{i} and U_{j} the transitions mapping.

(2.5) $$\varphi_{i,j} = (\varphi_{j} \circ \varphi_{i}^{-1}) : \varphi_{i}(U_{i} \cap U_{j}) \to \varphi_{j}(U_{i} \cap U_{j})$$

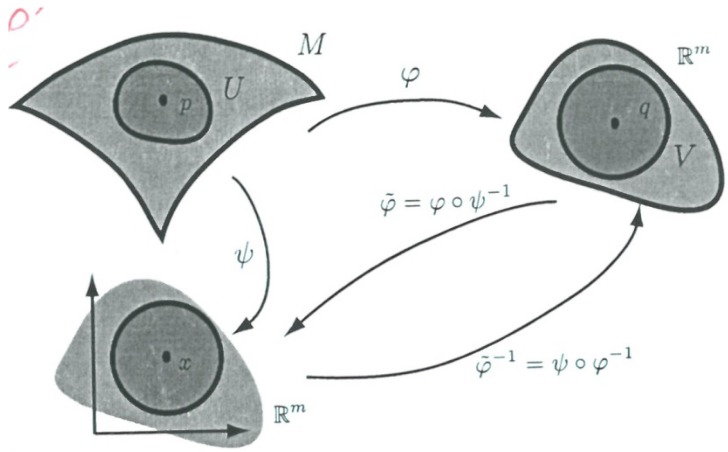
And as such are homeomorphisms between these open of R^{m} , for example the differentiability $(\varphi'' \circ \varphi^{-1})$ is achieved the mapping $(\varphi'' \circ (\tilde{\varphi})^{-1})$ and $(\tilde{\varphi} \circ \varphi^{-1})$ which are homeomorphisms since $(A \approx A'')$ by assumption this establishes the equivalence $(A \approx A'')$, for example let N and M be smooth manifolds n and m respecpectively , and let $f : N \to M$ be smooth mapping in local coordinates $f' = (\psi \circ f \circ \varphi^{-1}) : \varphi(U) \to \psi(V)$ Figurer (5) ,with respects charts (U, φ) and (V, ψ) , the rank of f at $p \in N$ is defined as the rank of f' at $\varphi(p)$ (i.e) $rk(f)_{p} = rk(J\, f')_{\varphi(p)}$ is the Jacobean of f at p this definition is independent of the chosen chart , the commutative diagram in that.

(2.6) $$f'' = (\psi' \circ \psi^{-1}) \circ \tilde{f} \circ (\varphi' \circ \varphi^{-1})^{-1}$$

Since $(\psi' \circ \psi^{-1})$ and $(\varphi' \circ \varphi^{-1})$ are homeomorphisms it easily follows that which show that our notion of rank is well defined $(J\, f'')_{x^{j}} = J(\psi' \circ \psi^{-1})_{y^{i}}\, J\, f'(\varphi' \circ \varphi^{-1})^{-1}$, if a map has constant rank for all $p \in N$ we simply write $rk(f)$, these are called constant rank mapping. The product two manifolds M_1 and M_2 be two C^k -manifolds of dimension n_1 and n_2 respectively the topological space $M_1 \times M_2$ are arbitral unions of sets of the form $U \times V$ where U is open in M_1 and V is open in M_2 , can be given the structure C^k manifolds of dimension n_1, n_2 by defining charts as follows for any charts M_1 on (V_j, ψ_j) on M_2 we declare that $(U_i \times V_j \ , \varphi_i \times \psi_j)$ is chart on $M_1 \times M_2$ where $\varphi_i \times \psi_j : U_i \times V_j \rightarrow R^{(n_1+n_2)}$ is defined so that.

(2,7) $$\varphi_i \times \psi_j(p,q) = (\varphi_i(p), \psi_j(q)) \text{ for all } (p,q) \in U_i \times V_j$$

A given a C^k n-atlas, A on M for any other chart (U,φ) we say that (U,φ) is compatible with the atlas A if every map $(\varphi_i \circ \varphi^{-1})$ and $(\varphi \circ \varphi_i^{-1})$ is C^k whenever $U \cap U_i \neq 0$ the two



Figurer (2):coordinate diffeomorphisms $\tilde{\varphi} = \varphi \circ \psi^{-1}$ **and** $\tilde{\varphi}^{-1} = \psi \circ \varphi^{-1}$

atlases A and $\tilde{A}$ is compatible if every chart of one is compatible with other atlas see Figure (2)

A sub manifolds of others of R^n for instance S^2 is sub manifolds of R^3 it can be obtained as the image of map into R^3 or as the level set of function with domain R^3 we shall examine both methods below first to develop the basic concepts of the theory of Riemannian sub manifolds and then to use these concepts to derive a equantitive interpretation of curvature tensor , some basic definitions and terminology concerning sub manifolds, we define a tensor field called the second fundamental form which measures the way a sub manifold curves with the ambient manifold , for example X be a sub manifold of Y of $\pi : E \rightarrow X$ and $g : E_1 \rightarrow Y$ be two vector brindled and assume that E is compressible , let $f : E \rightarrow Y$ and $g : E_1 \rightarrow Y$ be two tubular neighborhoods of X in Y then there exists a C^{p-1} .

The smooth manifold , an n-dimensional manifolds is a set that looks like R^n . It is a union of subsets each of which may be equipped with a coordinate system with coordinates running over an open subset of R^n . Here is a precise definition.

Definition 2.6

Let M be a metric space we now define what is meant by the statement that M is an n-dimensional C^∞ manifold.

I. A chart on M is a pair (U,φ) with U an open subset of M and φ a homeomorphism a (1-1) onto, continuous function with continuous inverse from U to an open subset of R^n , think of φ as assigning coordinates to each point of U .

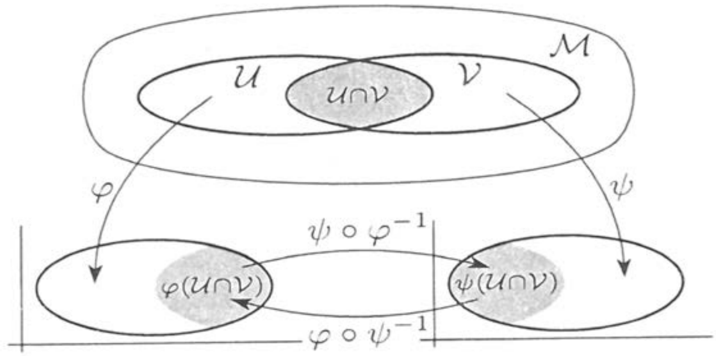
II. Two charts (U,φ) and (V,ψ) are said to be compatible if the transition functions . see Fig (1)

(3) $$\psi \circ \varphi^{-1} : \varphi(U \cap V) \subset R^n \rightarrow \psi(U \cap V) \subset R^n$$
$$\varphi \circ \psi^{-1} : \psi(U \cap V) \subset R^n \rightarrow \varphi(U \cap V) \subset R^n$$

Are C^∞ that is all partial derivatives of all orders of $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ exist and are continuous.

III. An atlas for M is a family $A = \{ (U_i, \varphi_i) : i \in I \}$ of charts on M such that $\{ U_i \}_{i \in I}$ is an open cover of M and such that every pair of charts in A are compatible . The index set I is completely arbitrary . It could consist of just a single index. It could consist of uncountable many indices . An atlas A is called maximal if every chart (U, φ) on M that is compatible with every chat of A .

IV. An n-dimensional manifold consists of a metric space M together with a maximal atlas A Figurer (3).



Figurer (3) : $(\varphi \circ \psi^{-1}) = (\psi^{-1} \circ \varphi)$

2.7 Example (open subset of R^n)

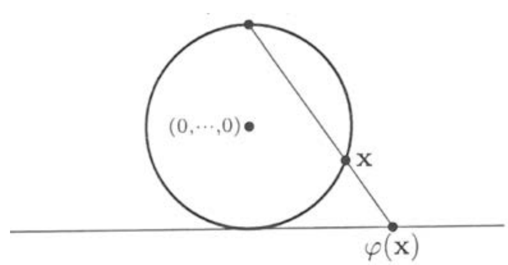
Let I_n be the identity map on R^n , then $\{ R^n, I_n \}$ is an atlas for R^n indeed , if U is any nonempty open subset of R^n , then $\{ U, I_n \}$ is an atlas for U so every open subset of R^n is naturally a C^∞ manifold.

2.8 Example (The n-sphere)

The n-space $S^n = \left\{ x = (x_1, \ldots, x_{n+1}) \in R^{n+1}, \left| x_1^2, \ldots, x_{n+1}^2 = 1 \right| \right\}$ is a manifold of dimension n when equipped with the atlas $A_1 = \left\{ (U_i, \varphi_i), (V_i, \psi_i), \left| 1 \le i \le n+1 \right| \right\}$ where for each $1 \le i \le n+1$.

(2.8) $$U_i = \left\{ (x_1, \ldots, x_{n+1}) \in S^n, x_1 \ge 0 \right\} \quad \varphi_i(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1})$$
$$V_i = \left\{ (x_1, \ldots, x_{n+1}) \in S^n, x_1 \le 0 \right\} \quad \psi_i(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1})$$

So both φ_i and ψ_i just discard the coordinate x_i they project onto R^n viewed as the hyper plane $x_i = 0$. A another possible atlas , compatible with A_1 is $A_2 = \{ (U, \varphi), (V, \psi) \}$ where the domains that $U = S^m \setminus \{0, \ldots, 0, 1\}$ and $V = S^n \setminus \{0, \ldots, 0, -1\}$ are the stereographic projection from the north and south poles, respectively , Both φ and ψ have range R^n plus an additional single point at infinity Figurer (4).



Figurer (4) :The n-sphere

Example 2.4 (Surfaces)

Any smooth n-dimensional R^{n+1} is an n-dimensional manifold. Roughly speaking a subset of R^{n+m} a an n-dimensional surface if , locally m of the $m+n$ coordinates of points on the surface are determined by the other n coordinates in a C^∞ way , For example , the unit circle S^1 is a one dimensional surface in R^2 . Near (0.1) a point $(x, y) \in R^2$ is on S^1 if and only if $y = \sqrt{1 - x^2}$ and near (-1.0) , (x, y) is on S^1 if and only if $y = -\sqrt{1 - x^2}$. The precise definition is that M is an n-dimensional surface in R^{n+m} if M is a subset of R^{n+m} with the property that for each $z = (z_1, \ldots, z_{n+m}) \in M$ there are a neighborhood U_z of z in

R^{n+m} , and n integers $1 \le J_1 \le j_2 \le \ldots \le j_{n+m}$, C^{∞} function $f_k(x_{j1},\ldots,x_{jn})$, $k \in \{1,\ldots,n+m\}/\{j_1,\ldots,j_n\}$ such that the point $x=(x_1,\ldots,x_{n+m}) \in U_z$. That is we may express the part of M that is near z as Figurer (5).

$x_{i1} = f_{i1}(x_{j1},x_{j2},\ldots,x_{jn}),\ x_{i2} = f_{i2}(x_{j1},x_{j2},\ldots,x_{jn}),\ \ldots,\ x_{im} = f_{im}(x_{j1},x_{j2},\ldots,x_{jn})$

Figurer (5) : as coordinates for R^2

Where there for some C^{∞} function $f_1,\ldots,f_m$. We many use $x_{j1},x_{j2},\ldots,x_{jn}$ as coordinates for R^2 in $M \cap U_z$ .Of course an atlas is $A=\{U_z \cap M, \varphi_z, | \ z \in M\}$, with $\varphi_z(x)=(x_{j1},\ldots,x_{jn})$ Equivalently, M is an n-dimensional surface in R^{n+m} if for each $z \in M$, there are a neighborhood U_z of z in R^{n+m} , and $m C^{\infty}$ functions $g_k : U_z \to R$ with the vector $\{\nabla_{gz}(z) | , 1 \le k \le m\}$ linearly independent such that the point $x \in U_z$ is in M if and only if $g_k(x)=0$ for all $1 \le k \le m$. To get from the implicit equations for M given by the g_k to the explicit equations for M given by the f_k one need only invoke (possible after renumbering of x) .

2.9 Theorem (Implicit Function)

Let $m,n \in N$ and let $U \subset R^{n+m}$ be an open set , let $g: U \to R^m$ be C^{∞} with $g(x_0,y_0)=0$ for some $x_0 \in R^n$, $y_0 \in R^m$ with $(x_0,y_0) \in U$. Assume that $\det[\frac{\partial g_i}{\partial y_j}(x_0,y_0)]_{1 \le i,j \le m} \ne 0$ then there exist open sets $V \subset R^{n+m}$ and $W \subset R^n$ with $(x_0,y_0) \in V$ such that , for each $x \in W$ there is a unique $(x\ ,y\ ) \in V$ with $g(x\ ,y\ )=0$ if the y above is denoted $f(x_0)=y_0$ and $g\ (x, f(x)\ )=0$ for all $x \in W$ the n-sphere S^n is the n-dimensional surface R^{n+1} given implicitly by equation $g(x_1,\ldots,x_{n+1}) = x_1^2 + \ldots + x_{n+1}^2 = 0$ in a neighborhood of , for example the northern hemisphere S^n is given explicitly by the equation $x_{n+1} = \sqrt{x_1^2 + \ldots + x_n^2}$ if you think of the set of all 3×3 real matrices as R^9 (because a 3×3 matrix has 9 matrix elements) then .

(2.9) $$SO(3)=\{3 \times 3\, real\ matrices\, R, R^t R = 1, \det R = 1\}$$

Is a 3-dimensional surface in R^9 , we shall look at it more closely Figurer (6) :

Figurer (6) : 3-dimensional surface in R^9

2.10 Example (A Torus)

The torus T^2 is the two dimensional surface $T^2 = \left\{ (x,y,z) \in R^3, (\sqrt{x^2+y^2}-1)^2 + z^2 = 1/4 \right\}$ in R^3 in cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta, z = 0$ the equation of the torus is $(r-1)^2 + z^2 = 1/4$ fix any $\theta, say\,\theta_0$. Recall that the set of all points in R^n that have $\theta = \theta_0$ is an open book , it is a hal-plane that starts at the z axis . The intersection of the tours with that half plane is circle of radius 1/2 centered on $r = 1, z = 0$ as φ runs form $0\, to\, 2\pi$, the point $r = 1 + 1/2\cos\varphi$ and $\theta = \theta_0$ runs over that circle. If we now run θ from $0\, to\, 2\pi$ the point $(x,y,z) = ((1+1/2\cos\varphi)\cos\theta, (1+1/2\sin\varphi)$ Runs over the whole torus . So we may build coordinate patches for T^2 using θ and φ with ranges $(0, 2\pi)$ or $(-\pi, \pi)$ as coordinates)

2.11 Definition

I. A function f from a manifold M to manifold N (it is traditional to omit the atlas from the notation) is said to be C^∞ at $m \in M$ if there exists a chart $\{U, \varphi\}$ for M and chart $\{V, \psi\}$ for N such that $m \in U$, $f(m) \in v$ and $\left(\psi \circ f \circ \varphi^{-1}\right)$ is C^∞ at $\varphi(m)$.

II. Tow manifold M and N are diffeomorphic if there exists a function $f : M \to N$ that is (1-1) and onto with N and f^{-1} on C^∞ everywhere. Then you should think of M and N as the same manifold with m and $f(m)$ being two names for same point , for each $m \in M$.

III. DIFFERENTIABLE MANIFOLDS AND TANGENT SPACE

In this section is defined tangent space to level surface γ be a curve is in R^n , $\gamma : t \to \left(\gamma^1(t), \gamma^2(t), ..., \gamma^n(t) \right)$ a curve can be described as vector valued function converse a vector valued function given curve , the tangent line at the point.

(3.1) $$\frac{d\gamma}{dt}(t) = \left(\frac{d\gamma^1}{dt} t_0 ...., \frac{d\gamma^n}{dt} t_0 \right)$$

we many k bout smooth curves that is curves with all continuous higher derivatives cons the level surface $f\left(x^1, x^2, ..., x^n \right) = c$ of a differentiable function f where x^i to $i-th$ coordinate the gradient vector of f at point $P = x^1(P), x^2(P), ..., x^n(P)$ is $\nabla f = \left(\frac{\partial f}{\partial x^1}, ..., \frac{\partial f}{\partial x^n} \right)$ is given a vector $u = (u^1, ..., u^n)$ the direction derivative $D_u f = \nabla f \cdot \overline{u} = \frac{\partial f}{\partial x^1} u^1 + ... + \frac{\partial f}{\partial x^n} u^n$, the point P on level surface $f\left(x^1, x^2, ..., x^n\right)$ the tangent is given by equation.

(3.2) $$\frac{\partial f}{\partial x^1}(P)(x^1 - x^1)(P) + ... + \frac{\partial f}{\partial x^n}(P)(x^n - x^n)(P) = 0$$

For the geometric views the tangent space shout consist of all tangent to smooth curves the point P , assume that is curve through $t = t_0$ is the level surface.

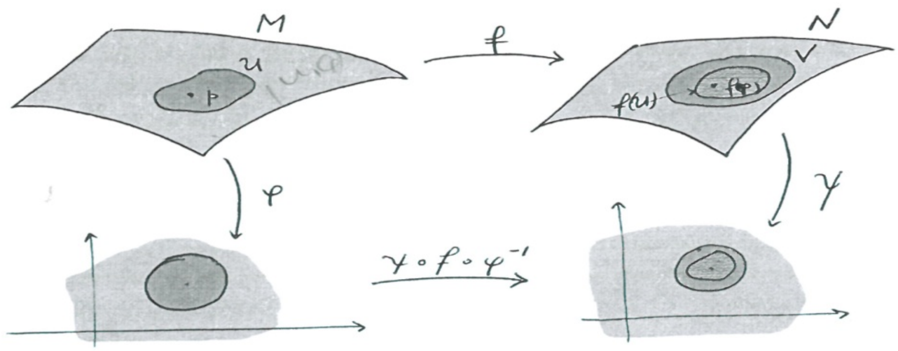
(3.3) $$f\left(x^1, x^2, ..., x^n\right) = c \,,\; f\left(\gamma^1(t), \gamma^2(t), ..., \gamma^n(t)\right) = c \text{ by}$$

taking derivatives on both $\frac{\partial f}{\partial x^1}(P)(\gamma'(t_0) + ... + \frac{\partial f}{\partial x^n}(P)\gamma^n(t)) = 0$ and so the tangent line of γ is really normal orthogonal to ∇f , where γ runs over all possible curves on the level surface through the point P . The surface M be a C^∞ manifold of dimension n with $k \geq 1$ the most intuitive to define tangent vectors is to use curves , $p \in M$ be any point on M and let $\gamma : \left]-\varepsilon, \varepsilon\right[\to M$ be a C^1 curve passing through p that is with $\gamma(M) = p$ unfortunately if M is not embedded in any R^N the derivative $\gamma'(M)$ does not make sense ,however for any chart (U, φ) at p the map $(\varphi \circ \gamma)$ at a C^1 curve in R^n and tangent vector $v = (\varphi \circ v)'(M)$ is will defined the trouble is that different curves the same v given a smooth mapping $f : N \to M$ we can define how tangent vectors in $T_p N$ are mapped to tangent vectors in $T_q M$ with (U, φ) choose charts $q = f(p)$ for $p \in N$ and (V, ψ) for $q \in M$ we define the tangent map or flash-forward of f as a given tangent vector.

(3.4) $$X_p = [\ \gamma\ ] \in T_p N \text{ and } d\ f_* : T_p M \quad , \quad f_*\left([\ \gamma\ ] = [\ f \circ \gamma\ ] \right)$$

A tangent vector at a point p in a manifold M is a derivation at p, just as for R^n the tangent at point p form a vector space $T_p(M)$ called the tangent space of M at p, we also write $T_p(M)$ a differential of map $f:N \to M$ be a C^∞ map between two manifolds at each point $p \in N$ the map F induce a linear map of tangent space called its differential $p, F_*: T_pN \to T_{F(p)}N$ as follows it $X_p \in T_pN$ then $F_*(X_p)$ is the tangent vector in $T_{F(p)}M$ defined see Figurer (7)

(3.5) $$\left(F_*\ (X_p)\ \right)f = X_p(\ f \circ F\ ) \in R \quad , \quad f \in C^\infty(M)$$



Figurer (7) : coordinate representation for f with $f(U) \subset V$

The tangent vectors given any C^∞ - manifold M of dimension n with $k \geq 1$ for any $p \in M$,tangent vector to M at p is any equivalence class of C^1 - curves through p on M modulo the equivalence relation defined in the set of all tangent vectors at p is denoted by T_pM we will show that T_pM is a vector space of dimension n of M . The tangent space T_pM is defined as the vector space spanned by the tangents at p to all curves passing through point p in the manifold M , and the cotangent T_p^*M of a manifold at $p \in M$ is defined as the dual vector space to the tangent space T_pM , we take the basis vectors $E_i = \left(\frac{\partial}{\partial x^i}\right)$ for T_pM and we write the basis vectors T_p^*M as the differential line elements $e^i = dx^i$ thus the inner product is given by.

(3.6) $$\left\langle \partial / \partial x, dx^i \right\rangle = \delta_i^j \ .$$

3.1 Definition

Let M_1 and M_2 be differentiable manifolds a mapping $\varphi: M_1 \to M_2$ is a differentiable if it is differentiable , objective and its inverse φ^{-1} is diffeomorphism if it is differentiable φ is said to be a local diffeomorphism at $p \in M$ if there exist neighborhoods U of p and V of $\varphi(p)$ such that $\varphi: U \to V$ is a diffeomorphism , the notion of diffeomorphism is the natural idea of equivalence between differentiable manifolds , its an immediate consequence of the chain rule that if $\varphi: M_1 \to M_2$ is a diffeomorphism then.

(3.7) $$d\,\varphi : T_pM_1 \to T_{\varphi(p)}M_2$$

Is an isomorphism for all $\varphi: M_1 \to M_2$ in particular , the dimensions of M_1 and M_2 are equal a local converse to this fact is the following $d\varphi: T_pM_1 \to T_{\varphi(p)}M_2$ is an isomorphism then φ is a local diffeomorphism at p from an immediate application of inverse function in R^n , for example be given a manifold structure again A mapping $f^{-1}: M \to N$ in this case the manifolds N and M are said to be homeomorphism , using charts (U, φ) and (V, ψ) for N and M respectively we can give a coordinate expression $\tilde{f}: M \to N$

3.2 Definition

Let M_1^{-1} and M_2^{-1} be differentiable manifolds and let $\varphi: M_1 \to M_2$ be differentiable mapping for every $p \in M_1$ and for each $v \in T_pM_1$ choose a differentiable curve $\alpha: (-\varepsilon, \varepsilon) \to M_1$ with $\alpha(M) = p$ and $\alpha'(0) = v$ take $\alpha \circ \beta = \beta$ the mapping $d\varphi_p: T_\varphi(p)M_2$ by given by $d\varphi(v) = \beta'(M)$ is line of α and $\varphi: M_1^{-1} \to M_2^{-1}$ be a differentiable mapping and at $p \in M_1$ be such $d\varphi: T_pM_1 \to T_\varphi M_2$ is an isomorphism then φ is a local homeomorphism .

3.3 Proposition

Let M_1^n and M_1^m be differentiable manifolds and let $\varphi : M_1 \to M_2$ be a differentiable mapping , for every $p \in M_1$ and for each $v \in T_p M_1$ choose a differentiable curve $\alpha : (-\varepsilon, \varepsilon) \to M_1$ with $\alpha(o) = p$, $\alpha'(o) = v$ take $\beta = \varphi \circ \alpha$ the mapping $d\varphi : T_p M_1 \to T_{\varphi(p)} M_2$ given by $d\varphi_p(v) = \beta'(o)$ is a linear mapping that dose not depend on the choice of α see Figurer (8)

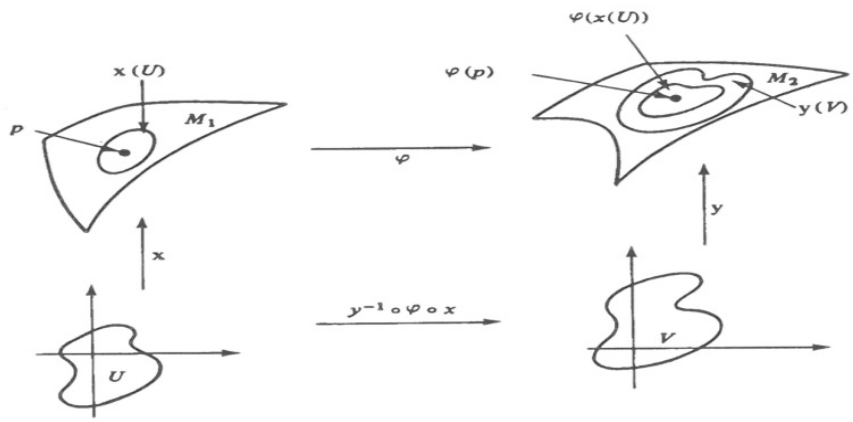


Figure (8) : $y^{-1} \circ \varphi \circ x$

3.5Theorem

The tangent bundle TM has a canonical differentiable structure making it into a smooth 2n-dimensional manifold , where N=dim. The charts identify any $U_p \in U(T_p M) \subseteq (TM)$ for an coordinate neighborhood $U \subseteq M$, with $U \times R^n$ that is hausdorff and second countable is called (The manifold of tangent vectors)

Definition 4.6

A smooth vectors fields on manifolds M is map $X : M \to TM$ such that

I. $X(P) \in T_p M$ for every G .

II. in every chart X is expressed as $a_i \, (\partial / \partial x_i)$ with coefficients $a_i(x)$ smooth functions of the local coordinates x_i .

4.7 Theorem

Suppose that on a smooth manifold M of dimension n there exist n vector fields $\left(x^{(1)}, x^{(2)}, ..., x^{(n)}\right)$ for a basis of $T_p M$ at every point p of M , then $T_p M$ is isomorphic to $M \times R^n$ m here isomorphic means that TM and $M \times R^n$ are homeomorphism as smooth manifolds and for every $p \in M$, the homeomorphism restricts to between the tangent space $T_p M$ and vector space $\{P_i \ \} \times R^n$.

Proof:

define $\pi : \vec{a} \in T_p M \subset TM$ on other hand , for any $M \times R^n$ for some $a_i \in R$ now define $\Phi : \vec{a} \in TM \to \left(\pi(s) : a_1, ..., a_n \in M \times R^n\right)$ is it clear from the construction and the hypotheses of theorem that Φ and Φ^{-1} are smooth using an arbitrary chart $\varphi : U \subseteq M \to R^n$ and corresponding chart

(3.10) $$\varphi T : \pi^{-1}(U) \subseteq TM \to R^n \times R^m$$

IV. INEGRATION SMOOTH MANIFOLD

We now onto integration .I shall explicitly define integrals over 0-dimensional , 1-dimensional and 2-dimensional regions of a two dimensional manifold and prove a generalization of Stokes theorem . I am restricting to low dimensions purely for pedagogical reason . The same ideas also work for high dimensions . Before getting into the details, here is a little motivational discussion. A curve , i.e a region that can be parameterized by function of real variable, is integral any finite union of , possibly disconnected, curves . We shall call this a 1-chain. We Start off integration of m-forms by considering m-forms R^m, a subset $D \subset R^m$ is called a domain of integration if D is bounded and ∂D has m-dimensional Lrbesgue measure $d\mu = dx_1, ..., dx_m$ equal to equal zero . In particular any finite union or intersection of open or closed rectangles is a domain of integration . Any bounded continuous function f on D is integral (i.e) $-\infty \prec \int_D f dx_1, ..., dx_m \prec \infty$ since $\Lambda^m(R^m) \cong R$ is a smooth function . For a given (bounded) domain of integration D we define .

(5.1) $$\int_D w = \int_D f(x_1, ..., x_m) dx_1 ... dx_m = \int_D f d\mu = \int_D w_x(e_1 .... e_m) d\mu$$

An m-form w is compactly supported if supp $(w) = cl\{ x \in R^m : w(x) \neq 0 \}$ is a compact set. The set of compactly supported m-form of R^m is denoted by $\Gamma_c^m(R^m)$, and is a linear subspace of $\Gamma_c^m(R^m)$. Similarly for any open set $U \subset R^m$ we can define $w \in \Gamma_c^m(R^m)$. Clearly $\Gamma_c^m(U) \subset \Gamma_c^m(R^m)$, and can be viewed as a linear subspace via zero extension to R^m. For any open set $U \subset R^m$ there exists a domain of integration D such that $D \supset D \supset \sup(w)$. For example let $U, V \in R^m$ be open sets $f : U \to V$ on orientation preserving diffeomorphism, and let $w \in \Gamma_c^m(V)$ then $\int_V w = \int_U f^* w$ if f for the domains D and E .we use coordinates $\{x_i\}$ and $\{y_i\}$ on D and D respectively . We start with $w = g(y_1, ..., y_m) dy^1 \wedge .... \wedge dy^m$. Using the change of variables formula for integrals and the pullback formula , we obtain .

(5.2) $$\int_E w = \int_E g(y) dy_1 .... dy_m = \int_D (f \circ g)(x) \det(J \tilde{f}_x) dx^1 \wedge ... \wedge dx^m = \int_D f^* w$$

One has to introduce a-sign in the orientation reversing case .

4.1 Theorem (Kelvin – Stokes)

(5.3) $$\int_D d\alpha = \int_{\partial D} i^* \alpha$$

For every $\alpha \in \Omega^{d-1}(M)$ where $i : \partial D \to M$ denotes the canonical (Moor prosaically , one says that $i^* \alpha$ is the restriction of α to ∂D) the attentive reader should have been worrying both integral above need some orientation to be defined . So we should add that the manifold M is oriented (or at least has a chosen local orientation covering at least D) then the basic ∂D inherits a canonical orientation from that of M , given geometrically by the inner side of D , and analytically by asking that dx_1 (locally) be used to orient the to normal directions to ∂D which will together with only one orientation to ∂D to produce the given orientation of M Figure (8) .

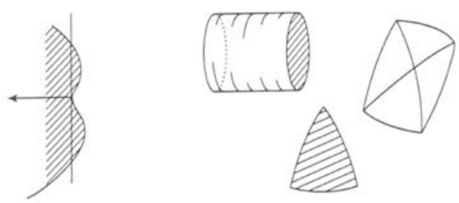


Figure (8) :domains with reasonable singularities

4.1 Definition (0-dimensional Integration)

I. A 0-form is a function $f : M \to C$.

II. A 0-chain is an expression of form $(n_1 P_1 + ... + n_k P_k)$ with $(P_1, ..., P_k)$ distinct points of M and $(n_1, ..., n_k) \in Z$. (c) If F is a 0-form and $(n_1 P_1 + ... + n_k P_k)$ is a 0-chain , then we define the integral.

(5.4) $$\int_{n_1 p_1 + ... + n_k p_k} F = n_1(P_1) + ... + n_k F(P_k)$$

4.2 Definition (1-dimensional Integration)

I. A 1-form w is a rule which assigns to each coordinate chart $\{U, \xi = (x, y)\}$ a pair (f, g) of com (f, g) complex valued functions on $\xi(U)$ in a coordinate manner to be defined in $w_{\{U,\xi\}} = f\,dx + g\,dy$ to indicate that w assigns the pair to the chart

$\{U,\xi\}$. That w is coordinate invariant means that – If $\{U,\xi\}$ and $\{\tilde{U},\tilde{\xi}\}$ are tow charts with $U\cap\tilde{U}\neq 0$ - If w assigns to $\{U,\xi\}$ the pair of functions (f,g) and assigns to $\{\tilde{U},\tilde{\xi}\}$ the pair of function $(\tilde{f},\tilde{g})$.

II. If the transition function $\{\tilde{\xi},\tilde{\xi}^{-1}\}$ from $\tilde{\zeta}(U\cap\tilde{U})\subset R^2$ to $\zeta(U\cap\tilde{U})\subset R^2$ is $(\tilde{x}(x,y),\tilde{y}(x,y))$ then.

(5.5)
$$f(x,y)=\tilde{f}(\tilde{x}(x,y),\tilde{y}(x,y))\frac{\partial\tilde{x}}{\partial x}(x,y)+\tilde{g}(\tilde{x}(x,y),\tilde{y}(x,y))\frac{\partial\tilde{y}}{\partial x}(x,y)$$
$$g(x,y)=\tilde{f}(\tilde{x}(x,y),\tilde{y}(x,y))\frac{\partial\tilde{x}}{\partial x}(x,y)+\tilde{g}(\tilde{x}(x,y),\tilde{y}(x,y))\frac{\partial\tilde{y}}{\partial y}(x,y)$$

III. If w is a 1-form and $(n_1C_1+....+n_kK_k)$ is a 1-chain then we define integral

(5.6)
$$\int_{n_1C_1+....+n_kC_k} w = n_1\int_{C_1} w+....+n_k\int_{C_k} w$$

IV. Addition of 1-form and multiplication of a 1-form by a function on M are defined as follows , let $\alpha : M\to C$ and let $\{U,\zeta=(x,y)\}$ be a coordinate chart for M . If $w_1|_{\{U,\xi\}}=f_1dx+g_1dy$ and $w_2|_{\{U,\xi\}}=f_2dx+g_2dy$ then .

(5.7)
$$w_1+w_2|_{\{U,\xi\}}=(f_1+f_2)\,dx+(g_1+g_2)\,dy$$
$$\alpha w_1|_{\{U,\xi\}}=\left(\alpha\circ\zeta^{-1}f_1\right)dx+\left(\alpha\circ\xi^{-1}g_1\right)dy$$

4.3 Definition (2-dimensional Integrals)

I. A 2-form Ω is a rule which assigns to each chart $\{U,\xi\}$ a function f on $\xi(U)$ such that $\Omega|_{\{U,\xi\}}=fdx\wedge dy$ is invariant under coordinate transformations . This means that .

II. If $\{U,\xi\}$ and $\{\tilde{U},\tilde{\xi}\}$ are two charts with $U\cap\tilde{U}\neq 0$ - If Ω assigns $\{U,\xi\}$ the function f and assigns $\{\tilde{U},\tilde{\xi}\}$ the function $\tilde{f}$ - If the transition function $\xi\circ\zeta^{-1}$ from $\xi(U\cap\tilde{U})\subset R^2$ to $\tilde{\xi}(U\cap\tilde{U})\subset R^2$ is $(\tilde{x}(x,y),\tilde{y}(x,y))$ then .

(5.8)
$$f(x,y)=\tilde{f}(\tilde{x}(x,y),\tilde{y}(x,y))\left[\frac{\partial\tilde{x}}{\partial x}(x,y)+\frac{\partial\tilde{y}}{\partial y}(x,y)-\frac{\partial\tilde{x}}{\partial y}(x,y)\frac{\partial\tilde{y}}{\partial x}(x,y)\right]$$

$Q^2=\{(x,y)\in R^2, x,y\geq 0, x+y\leq 1\}$ a surface is map $D:Q^2\to M$ 2-chain is an expression of the from $(n_1D_1+....+n_kD_k)$ with $(D_1+....+D_k)$ surfaces and $(n_1+....+n_k)$ surfaces and $(n_1+....+n_k)\in Z$.

III. Let $\{U,\xi=(x,y)\}$ be a chart and let $\Omega|_{U,\xi}=f(x,y)\,dx\wedge dy$ if $D:Q^2\to U\subset M$ is a surface with range in U then we define the integral .

(5.9)
$$\int_D\Omega=\iint_{Q^2} f(\xi(D(s,t)))\left[\frac{\partial}{\partial s}x(D(s,t))\frac{\partial}{\partial s}y(D(s,t))-\frac{\partial}{\partial t}x(D(s,t))\frac{\partial}{\partial s}y(D(s,t))\right]ds\,dt$$

If D does not have rang in a single chart , split it up into a finite number of pieces, each with range in a single chart. This can always be done , since the range of D is always compact . The answer is independent of chart (s) .

IV. If Ω is a 2-form and $(n_1D_1+...+n_kD_k)$ is a 2-chain , then we define the integral.

(5.10)
$$\int_{n_1D_1+....+n_kD_k}\Omega = n_1\int_{D_1}\Omega+......+n_k\int_{D_k}\Omega$$

4.4 Definition (n-dimensional Integrals)

The integrals of n-forms w on M ,we first assume that w is a n-form supported in an orientation compatible coordinate chart $\{\varphi,U,V\}$ so that there is a function $f(x^1,...,x^n)$ supported in U such that $w=f(x^1,...,x^n)dx^1\wedge....\wedge dx^n$ we define $\int_U w=\int_V f(x^1,.....,x^n)dx^1,...,dx^n$ where the right hand side is the Lebesgue integral on $V\subset R^n$. To integrate a general n-form w on M , we take a locally finite cover $\{U_\alpha\}$ of M that consists of orientation-compatible coordinate charts. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$ .Now since each ρ_α is supported in U_α each $\rho_\alpha w$ is supported U_α also .We define

(5.11)
$$\int_M w=\sum_\alpha\int_{U_\alpha}\rho_\alpha w$$

We say that w is integral if the right hand side converges. One need to check that the definition above is independent of choice of orientation compatible coordinate charts , and is independent of choice of partition of unity , so that the integral is well-defined .

4.5 Theorem

The expression (6) is independent of choice of U_α and the choice of ρ_α .

Proof :

We first show that $\int_U w = \int_V f(x^1,.....,x^n)dx^1,...,dx^n$ is well-defined , i.e w is supported in U and if $\{x_\alpha^i\}$ and $\{x_\beta^i\}$ are $w = f_\alpha dx_\alpha^1 \wedge ... \wedge dx_\alpha^n = f_\beta dx_\beta^1 \wedge .... \wedge dx_\beta^n$ then .

$\int_{V_\alpha} f_\alpha dx_\alpha^1,....,dx_\alpha^n = \int_{V_\beta} f_\beta dx_\beta^1,...,dx_\beta^n$ then $dx_\beta^1,....,dx_\beta^n = \det(d\varphi_{\alpha\beta})dx_\beta^1,...,dx_\beta^n$ implies that

$f_\alpha = \det(d\varphi_{\alpha\beta})\, f_\beta$ on the other hand side, the change of variable formula in R^n reads

(5.12) $$\int_{V_\beta} f dx_\beta^1,....,dx_\beta^n = \int_\alpha f \det(d\varphi_{\alpha\beta})dx_\alpha^1,...,dx_\alpha^n$$

So that desired formula follows form the fact $\det(d\varphi_{\alpha\beta}) \succ 0$ since U_α and U_β are orientation compatible . Well-defined ,we suppose U_α and U_β are two locally finite cover of M consisting of orientation-compatible charts, and ρ_α and ρ_β are partitions of unity subordinate to U_α and U_β respectively . We consider a new cover $U_\beta \cap U_\alpha$ with new partition of unity ρ_α , ρ_β it is enough to prove $\Sigma_\alpha \rho_\alpha w = \int_{U_\beta} (\Sigma_\beta \rho_\beta)\rho_\alpha w = \sum_\beta \int_{U_0 \cap U_\beta} \rho_\beta . \rho_\alpha w$ obviously the integral defined above is linear $\int_M (aw + b\eta) = a\int_M w + b\int_M \eta$. Now M , N are both oriented manifolds, with volume forms $\eta_1 \eta_2$ respectively .

4.6 Definition

A smooth map $f : M \to N$ is said to be orientation-preserving if $f^*\eta_2$ is a volume form on M that defines the same orientation as η_1 does .

4.7 Theorem

Let M be compact manifold and α , β $\int_M f^* w = \int_N w$.

Proof :

It is enough to prove this in local charts tow volume forms then there exist a in which case this is merely change of variable formula in R^n .

V. STOKES THEOREM

Now let M be a compact oriented n-dimensional manifold with boundary ∂M the induced orientation as follows if M is described locally near the boundary by $x^n \geq 0$ then the positive orientation on ∂M is the one corresponding to $\delta dx^1 \wedge .... \wedge dx^{n-1}$ where $\delta = \pm 1$ is determined by the relation .

(6.1) $$d\left(-x^n\right) \wedge dx^1 \wedge .... \wedge dx^{n-1} = \delta dx^1 \wedge .... \wedge dx^n$$

5.1 Theorem (Stokes)

Let M be an oriented n-dimensional manifolds $U \subset M$ open with smooth boundary $w \in \Omega^{-1}(M)$ .Assume that $\overline{U}$ is compact or that w has compact support then

$$\int_U dw = \int_{\partial U} w$$

Proof :

We cover U by charts $\varphi_j : U_j \to V_j$, such that meeting the boundary ∂U are o the form required in choose a subordinate partition of unity ψ_j with compact supports , we can re-number the charts to have the same index set , letting φ_j denote a chart whose domains contains supp ψ_j we may ψ_j such that its support meets $\overline{U}$ and the support of w (recall that the family of supports of the ψ_j is locally finite) . We then have .

(6.2) $$\int_U dw = \int_U d(\sum_j \psi_j w) = \sum_j \int_U d(\psi_j w) = \sum_j \int_{V_j} d(\psi_j w)\varphi_j = \sum_j \int_{V_j} d(\psi_j w)\varphi_j$$

Not that the sum is finite, so there is no problem in interchanging it with the integration . Now if $U_j \cap \partial U = 0$ then (by an argument similar to that used in the proof of the "Baby Stokes" $\int_{V_j} d(\psi_j w)\phi_j = 0$ In the other case ,we find

by $\int_{V_j} d(\psi_j w)\phi_j = \int_{V_j^{/}} (\psi_j w)\varphi_j^{/}$ where $\varphi_j^{/} : U_j \cap \partial U \to V_j^{/}$ is the chart of ∂U obtained by restricting φ_j and projecting to the last n-1 coordinates . Let $\sum_j^{/}$ denote the sum restricted to those j such that $U_j \cap \partial U = 0$ then we find.

(6.3) $$\int_U dw = \int_U d(\sum_j \psi_j w) = \sum_j \int_U d(\psi_j w) = \sum_j \int_{V_j} d(\psi_j w)\varphi_j = \sum_j \int_{V_j} d(\psi_j w)\varphi_j$$

Note that on $\partial U, \sum_j^{/} \psi_j = 1$.

Let us see what Stokes Theorem tells us about integration in R^3 (or in R^n) . First we need to find out what integrals of k-forms on R^3 correspond to .

5.2 Definition (0-Forms)

A 0-form is just a function $f \in C^\infty(R^n) = \Omega^0(R^n)$ we can integrate it over a 0-dimensional subset , which is just a finite collection of oriented points on a single point $p \in R^n$ we have

(6.4) $$\int_{\pm p} f = \pm f(p)$$

where the sign denotes the orientation of the point (and is not to the coordinates of $P!$).

5.3 Definition (1-Forms)

A 1-form on R^n has the shape $w = f_1(x)dx_1 + f_2(x)dx_2 + ... + f_n(x)dx_n$ we can identify w with the vector field $F = (f_1, ..., f_n)^T$ if $\gamma : [a,b] \to R^n$ is a curve then .

(6.5) $$\int_\gamma w = \int_a^b (f_1(\gamma(t)\gamma'(t) + ... + f_n(\gamma(t))\gamma_n'(t))dt = \int_a^b F(\gamma(t)\gamma'(t)dt = \int_\gamma F.v|dx|$$

The latter denoting the unrented integral v denotes the unit tangent vector in direction of the orientation of the curve. For this we assume that $\gamma'(t)$ does not vanish then $\gamma'(t) = v\ (\gamma(t) \parallel \gamma'(t) \parallel$ and $\gamma'(t)$ is the factor $\sqrt{\det(D\gamma^T D\gamma)}$ in the definition of the volume integral (which can be carried over to immersed manifolds i.e subset parameterized by open subset of R^k such that the derivative of the parameterization (like γ here) has maximal rank k every where) . Not that $F.v$ given the tangential component of the vector field along curve.

5.4 Definition ((n-1)-Forms)

(n-1)-forms on R^n looks like this :

$$\eta = u_1(x)dx_2 \wedge .... \wedge dx_n - u_2(x)dx_1 \wedge dx_3 \wedge .... \wedge dx_n + ... + (-1)^{n-1}u_n(x)dx_1 \wedge ... \wedge dx_{n-1}$$
$$= \sum_{i=1}^n (-1)^{i-1}u_i(x)dx_1 \wedge .... \wedge dx_{i-1} \wedge dx_{i+1} \wedge .... \wedge dx_n$$

We can identify η with the vector field $U = (u_1, ..., u_n)^T$ if $\theta : W \to R^n$ with $W \subset R^{n-1}$ open parameterizes a hyper surface in R^n then we have :

(5.6) $$\int_\phi \eta = \int_W \phi^* \eta = \int_W \left(\sum_{i=1}^n (-1)^{i-1} u_i(\phi(x))d\phi_1 \wedge .... \wedge d\phi_{i+1} \wedge .... \wedge d\phi_n\right)$$
$$= \int_W \det\left(U(\phi(x), \frac{\partial \phi}{\partial x_1}, ..., \frac{\partial \phi}{\partial x_{n-1}} |d^{n-1}x|\right) = \int_\phi U.n|d^{n-1}x|$$

Here , n is the normal vector to the tangent space of the hyper surface that :

$$n, (U(\phi(x))\ , \frac{\partial \phi}{\partial x_1}, ..., \frac{\partial \phi}{\partial x_{n-1}} = U(\phi(x)).n(\phi(x)).n(\phi(x)\sqrt{\det(D\phi_x^T D\phi_x)}$$

5.5 Definition (n-forms)

An n-form on R^n is given by $r(x)dx_1 \wedge dx_2 \wedge .... \wedge dx_n$ for an open subset $U \subset R^n$, we then simply have.

(6.7) $$\int_U r(x)\ (dx_1 \wedge dx_2 \wedge .... \wedge dx_n\ ) = \int_U r(x) \left|\ d^m x\ \right|$$

Now we can interpret Stokes Theorem in these cases .

5.6 Definition (Stokes for Curves and 0-Forms)

Let $\gamma : [a,b] \to R^n$ be a curve , $f \in C^\infty(R^n)$ then $\int_\gamma d\ f\ =\ f(\gamma(b)) - f(\gamma(a))$ and

(6.8) $$\int_\gamma df\ = \int_\gamma \nabla f\ .v|dx| = \int_\gamma D_v\ f|dx|$$

Is the integral of directional derivative of f in direction of the unit tangent vector of the curve . This generalizes the Fundamental of Calculus to line integrals.

5.7 Definition (Stokes for open Subset (n-1)-form)

Let $U \subset R^n$ be an open subset with smooth boundary ∂U and assume say that is $\overline{U}$ compact . Let $\eta \in \Omega^{n-1}(R^n)$, corresponding to the vector field U as above , then .

$$d\eta = \Sigma_{i=1}^{n} \frac{\partial u_i}{\partial x_i} dx_1 \wedge .... \wedge dx_n = \nabla .U dx_1 \wedge .... \wedge dx_n$$

$\nabla .U$ is the divergence of U .we then obtain what is known as the Divergence Theorem (also called Gauss Theorem) .

(6.9) $$\int_U \nabla .U \left|\ d^m x\ \right| = \int_{\partial U} U.\ n \left| d^{m-1} x\ \right|$$

Where n is the outer unit normal vector . This says that the total flow out of the set U is the same as the total divergence of the vector field inside U , this justifies the interpretation of the divergence as the amount of flow that is " generated " at a point .

5.8 Green's Theorem

This is the special case $n = 2$ of the preceding incarnation (or also the planar case of the following) . If $S \subset R^2$ is open set and bounded with sufficiently nice boundary curve ∂S (oriented counter – clockwise) and if $f, g \in C^\infty(R^n)$ then .

(6.10) $$\int_s \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \left|\ dx\, dy\ \right| = \int_{\partial S} \left(f\ dx + g\ dy\ \right)$$

5.9 Example

Let $U \subset R^2$ be the upper semi-disk of radius R .To find $\int_{\partial U} (xdx + 2xydy)$ we can parameterize the two parts of the boundary and then have to integral some polynomial in $\sin t\ and \cos t$. Alternatively , we can use Green's Theorem and get.

(6.11) $$\int_{\partial U} (x^2 dx + 2xydy) = \int_U 2y|dx\, dy\ | = \int_{-R}^{R} \int_0^{\sqrt{R^2 - x^2}} 2ydydx = \int_{-R}^{R} \left(R^2 - x^2 \right) dx = \frac{4}{3} R^3$$

5.10 Definition (Stokes for Surfaces and 1-Forms on R^3)

Let $w \in \Omega^1(R^2)$ and let $\phi : W \to R^3$ be a parameterized surface (with $W \subset R^2$) . Let F be the vector field corresponding to w . If U is the vector field corresponding to dw , then we have $U = \nabla \times F$ i.e U is the curl of F. From Stokes Theorem we obtain what is in fact the original result of Stokes .

(6.12) $$\int_S (\nabla \times F).n \left|\ d^2 x \right| = \int_{\partial S} F.v \left|\ dx\ \right|$$

Here S is the oriented surface parameterized by ϕ , and ∂S is its boundary curve, oriented such that we move around S counter-clockwise when looking from the side in which the normal vector n points. This leads to interpretation of the curl as given the amount of rotation of the flow around a given axis (here , n).

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IV. M be an oriented n-dimensional manifolds $U \subset M$ open with smooth boundary $w \in \Omega^{-1}(M)$ Assume that $\overline{U}$ is compact or that w solutions by stokes theorem on domain R^n is $\int_U dw = \int_{\partial U} w$

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APPENDIX

Appendixes, if needed, appear before the acknowledgment.

ACKNOWLEDGMENT

The preferred spelling of the word “acknowledgment” in American English is without an “e” after the “g.” Use the singular heading even if you have many acknowledgments.

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