Sigma Elements in Lattice Modules

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Abstract- Let L be a compactly generated multiplicative lattice with 1 compact in which every finite product of compact elements is compact and M be a module over L which is also a compactly generated in which the largest element I_M is compact. In this paper we define A_p , A_{FM} in the lattice module M and obtain many properties where p is prime element of L . Finally we define σ- element in a lattice module M and obtain it's properties.

Index Terms- Prime element, Primary element, Lattice Modules.
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I. Introduction

A multiplicative lattice L is a complete lattice provided with commutative, associative and join distributive multiplication in which the largest element 1 acts as a multiplicative identity. An element $a \in L$ is called proper if a < 1.A proper element p of L is said to be prime if $ab \leq p$ implies $a \leq p$ or $b \leq p$. If $a \in L, b \in L$, (a : b) is the join of all elements c in L such that $cb \leq a$. A proper element p of L is said to be primary if $ab \leq p$ implies $a \leq p$ or $b^n \leq p$ for some positive integer n. If $a \in L$, then $\sqrt{a} = \bigvee\{x \in L_* \mid x^n \leq a, n \in Z_+\}$. An element $a \in L$ is called a radical element if $a = \sqrt{a}$. An element $a \in L$ is called compact if $a \leq \bigvee_\alpha b_\alpha$ implies $a \leq b_{\alpha_1} \vee b_{\alpha_2} \dots \vee b_{\alpha_n}$ for some finite subset$\{\alpha_1, \alpha_2 \dots \alpha_n\}$. Throughout this paper, L denotes a compactly generated multiplicative lattice with 1 compact in which every finite product of compact element is compact. We shall denote byL_*, the set of compact elements of L. A nonempty subset of L_* is called a filter if the following conditions are satisfied.

i) $x, y \in F$ implies $xy \in F$

ii) $x \in F, x \leq y$ implies$y \in F$.

Let F(L_*) denote a set of all filters of L_*. For a nonempty subset $F_\alpha \subseteq F(L_*)$, define $\Cup F_\alpha = \{x \geq f_1, f_2 \dots f_n, f_i \in F_{\alpha_i}, \ for\ some\ i = 1,2 \dots n\}$. Then it is observed that, $< F(L_*), \Cup, \cap >$ is a complete distributive lattice with $\Cup$ as the supremum and the set theoretic $\cap$ as infrimum. For $a \in L_*$ the smallest filter containing a is denoted by [a) and it is given by [a) = { $x \in L_* \mid x \geq a^n$, for some non-negative integer n}. For a filter F ∈ F(L_*) we denote, 0_F= $\bigvee\{x \in L_* \mid xs = 0$, for some s ∈ F}. If p is a prime element of L then Fp = {x ∈ L_*| x $\nleq$ p} is a filter in F(L_*). An element e ∈ L is called meet principal if for all a; b ∈ L, a ∧ be = ((a : e) ∧ b) e. An element e ∈ L is called join principal if for all a; b ∈ L, (ae ∨ b) : e = (b : e) ∨ a. An element e ∈ L is called principal if e is both meet and join principal. A multiplicative lattice L is said to be principally generated (PG) if every element of L is a join of principal elements of L.

Let M be a complete lattice and L be a multiplicative lattice. Then M is called an L- module or module over L if there is a multiplication between elements of L and M written as aB, where a ∈ L and B ∈ M which satisfies the following properties:

i) $(\bigvee_\alpha a_\alpha)$A= $\bigvee_\alpha a_\alpha$A , ∀a_α ∈ L , A ∈ M

ii) a $(\bigvee_\alpha A_\alpha)$= $\bigvee_\alpha aA_\alpha$, ∀a ∈ L; A_α ∈ M.

iii)(ab)A = a(bA) ∀a; b ∈ L; A ∈ M

iv)1B = B

v) 0B = 0_M, ∀a; a_α; b ∈ L and A; A_α ∈ M, where 1 is the supremum of L and 0 is the infimum of L. We denote by 0_M and I_M the least and the greatest element of M. Elements of L will generally be denoted by a, b, c ... and elements of M will generally be denoted by A,B,C....

Let M be an L- module. If N ∈ M and a ∈ L then (N : a) = $\bigvee\{X \in M | aX \leq N\}$. If A, B ∈M, then (A : B) = ∨{x ∈ L = xB ≤ A}. An element A ∈ M is said to be a compact if A ≤ $\bigvee_\alpha B_\alpha$ implies A ≤ $B_{\alpha_1} \vee B_{\alpha_2} ... \vee B_{\alpha_n}$ for some finite subset { $B_{\alpha_1}, B_{\alpha_2} \dots B_{\alpha_n}$} . A set of compact elements of M will be denoted by M_*. An L- module M is called a multiplication L-module if for every element N ∈ M there exists an element a ∈ L such that N = aI_M see [3]. In this paper a lattice module M will be a multiplication lattice module, which is compactly generated with the largest element I_M compact in which product of a compact element of L and a compact of compact element of M will be compact. A proper element N of M is said to be prime if aX ≤ N implies X ≤ N or aI_M ≤ N that is a ≤ (N : I_M) for every a ∈ L, A ∈ M. If N is a prime element of M then (N : I_M) is a prime element of L. [3]. An element N < I_M in M is said to be primary if aX ≤ N implies X ≤ N or $a^n I_M$ ≤ N that is a^n ≤ (N : I_M) for some integer n. An element N of M is called a radical element if (N : I_M) = $\sqrt{N : I_M}$. If aN = 0_M implies a = 0 or N = 0_M for any a ∈ L and N ∈ M then M is called torsion free L- module. If (0_M : I_M) = 0 then M is called a torsion free faithful L - module. In this paper a module M will be a torsion free faithful, multiplication PG- lattice L- module. For all these definitions and any other undefined term one can refer [1],[2].

II. Properties of Ap, $A_{FM}, *_{VM}$.

Let M be a module over a multiplicative lattice L. Let A ∈ M and p be a prime element of L. We define, A_p = ∨{xI_M | x ∈ L_; yxI_M ≤ A, y ≰ p} and A_{FM} = ∨{X ∈ M_* | sX ≤ A; for some s ∈ F ∈ F_*}. $*_{VM}$ = {V_{FM} | F ∈ F(L_*); F ⋂[0, (V : IM)] = ∅ }.

Theorem (2.1) Let A ∈ M. Then A_p = ∨ {(A : y) | y ≰ p; y ∈ L_*}= ∨{(A : y) | y ≰ p, y ∈ L}.
Proof:- Let z ∈ L_* such that zI_M ≤ A_p. Then yzI_M ≤ A, y ≰ p. Hence zI_M ≤ (A: y), y ∈ L implies A_p ≤ ∨{(A : y) | y ∈ L, y ≰ p}. Let Z = zI_M ∈ M_* such that zI_M ≤ ∨{(A : y) | y ∈ L, y ≰ p}: Then Z = zI_M ≤ (A : y_1) ∨(A : y_2) ∨… ∨ (A : y_n); y_i ≰ p; i = 1, 2,3... n. Let $y_i = \bigvee_{\alpha_i} y_{\alpha_i}$, y_{α_i} ∈ L_*. Then y_i ≰ p implies y_{α_i}≰p, for some α_i, y_{α_i} ≤ y_i. But y_{α_i} ≤ y_i implies (A : y_i) ≤ (A : y_{α_i}) and zI_M ≤ (A : y_{α_1}) ∨ (A : y_{α_2}) ∨ … ∨ (A : y_{α_n}). Let y = $y_1 y_2 ... y_n$. Then y ≤ y_{α_1}, y ≤ y_{α_2}, … ,y ≤ y_{α_n} and $y_{\alpha_1} y_{\alpha_2}$ ... y_{α_n} = y ≰ p, (y ∈ L_*). Also y ≤ y_{α_i} implies (A : y_{α_i}) ≤ (A : y). Therefore zI_M ≤ (A : y), y ≰ p. But yzI_M ≤ A, y ≰ p implies zI_M ≤ A_p. So we have, A_p = ∨ {(A : y) | y ∈ L, y ≰ p}.
To prove ∨{(A : y) | y ∈ L,y ≰ p} = ∨{(A : y) | y ≰ p, y ∈ L_*}. Let zI_M ≤ A_p. Then yzI_M ≤ A, y ≰ p. But y = ∨y_α, y_α is compact. Then y_α ≰ p for some y_α ∈ L_*, y_α ≤ y. Hence y_αzI_M ≤ A and zI_M ≤ (A : y_α), y_α ≰ p, y_α ∈ L_*. This shows that A_p ≤ ∨{(A : y) | y ∈ L_*,y ≰ p}. Let Z = zI_M ∈ M_* such that zI_M ≤ ∨{(A : y) | y ∈ L_*, y ≰ p}. Then zI_M ≤ (A : y_1) ∨ (A : y_2) ∨ … ∨ (A : y_n), y_i ∈ L_*, yi ≰ p. Let y = $y_1, y_2, ..., y_n$. Then y ∈ L_*, y ≤ y_i, and (A : y_i) ≤ (A : y) for all i, y ≰ p. Hence zI_M ≤ (A : y) and yzI_M ≤ A; y ≰ p, y ∈ L_* and zI_M ≤ A_p. Thus ∨{(A : y) | y ∈ L_*, y ≰ p} ≤ A_p and A_p = ∨{(A : y) | y≰ p, y ∈ L_*} = ∨{(A : y) | y ≰ p, y ∈ L} .
Theorem (2.2) A ≤ A_{FM} for any A ∈ M.
Proof:- By (Theorem 2.1 [7]), we have, A ≤ (A : y), y ∈ F. Therefore, A ≤ A_{FM}.
Theorem (2.3) If A ∈ M, $(A_{FM})_{FM}$= A_{FM}.
Proof:- (By Theorem 2.2) A_{FM} ≤ $(A_{FM})_{FM}$. Let Z ∈ M_* such that Z ≤ $(A_{FM})_{FM}$. Then sZ ≤ A_{FM}, for some s ∈ F. Hence rsZ ≤ A, for some r ∈ F. This gives Z ≤ A_{FM} where rs ∈ F. Consequently $(A_{FM})_{FM}$ ≤ A_{FM} and hence $(A_{FM})_{FM}$ = A_{FM}.
Theorem (2.4) Let A, B ∈ M. Then A ≤ B implies A_{FM} ≤ B_{FM}.
Proof:- A ≤ B implies (A : y) ≤ (B : y) , y ∈ F. So ∨ {(A : y), y ∈ F} ≤ ∨{(B : y), y ∈ F} and hence A_{FM} ≤ B_{FM}.
Theorem (2.5) Let F, G ∈ F(L_*). Then $(A_{FM})_{GM}$ = $(A_{GM})_{FM}$.
Proof:- Let X ∈ M_* such that X ≤ $(A_{FM})_{GM}$. Then tX ≤ A_{FM}, t ∈G and stX ≤ A, for some s ∈F. This shows that sX ≤ A_{GM}, for some s ∈ F and hence X ≤ $(A_{GM})_{FM}$. Hence $(A_{FM})_{GM}$ ≤ $(A_{GM})_{FM}$. Similarly $(A_{GM})_{FM}$ ≤ $(A_{FM})_{GM}$ which shows that $(A_{FM})_{GM}$ = $(A_{GM})_{FM}$.

The following result is proved in [7]

Theorem (2.6) For X ∈ M_*, X ≤ A_{FM} if and only if yX ≤ A, for some y ∈ F.

Theorem (2.7) If L is local, with unique maximal element m, then A_m = A.

Proof:-Since m is a maximal, it is prime and $A_m = \bigvee_{x \nleq m}\{(A : x)\}$. But (A : 1) = ∨{X ∈ M | 1. X ≤ A} = A and $A_m = \bigvee_{x \nleq m}\{(A : x)\}$ = (A : 1) = A.

Theorem (2.8) Let F ∈ F(L_*) and A,B ∈ M. Then $A_{FM} \wedge B_{FM} = (A \wedge B)_{FM}$.

Proof:- Let X ∈ M_* be such that X ≤ $A_{FM} \wedge B_{FM}$. Then X ≤ A_{FM} and X ≤ B_{FM}. Hence rX ≤ A and sX ≤ B for some r,s ∈ F. This gives, (r ∧ s)X ≤ rX and (r ∧ s)X ≤ sX. So (r ∧ s)X ≤ rX ∧ sX ≤ (A ∧ B),where (r ∧ s) ∈ F. This shows that X ≤ $(A \wedge B)_{FM}$ and $A_{FM} \wedge B_{FM} \leq (A \wedge B)_{FM}$. Conversely, let X ≤ $(A \wedge B)_{FM}$. So rX ≤ (A ∧ B), for some r ∈ F which gives rX ≤ A and sX ≤ B. Hence X ≤ A_{FM}, X ≤ B_{FM} and hence X ≤ $A_{FM} \wedge B_{FM}$. Therefore $(A \wedge B)_{FM} \leq A_{FM} \wedge B_{FM}$ and $A_{FM} \wedge B_{FM} = (A \wedge B)_{FM}$.

Theorem (2.9) If A,B ∈ M, $A_{FM} \vee B_{FM} \leq (A \vee B)_{FM}$.

Proof:- As A ≤ A ∨ B and B ≤ A ∨ B, we have $A_{FM} \leq (A \vee B)_{FM}$, $B_{FM} \leq (A \vee B)_{FM}$, and hence $A_{FM} \vee B_{FM} \leq (A \vee B)_{FM}$.

Theorem (2.10) If A, B ∈ M, $(A_{FM} \vee B_{FM})_{FM} = (A \vee B)_{FM}$.

Proof:- Let X ≤ $(A \vee B)_{FM}$, where X ∈ M_*. Then rX ≤ (A ∨ B), for some r ∈ F. Therefore rX ≤ ($A_1 \vee B_1$), for some A_1 ≤ A, B_1 ≤ B, where $A_1, B_1 \in M_*$, but A_1 ≤ A and 1. A_1 ≤ A, 1 ∈ F implies $A_1 \leq A_{FM}$. Similarly $B_1 \leq B_{FM}$. So rX ≤ $A_{FM} \vee B_{FM}$ implies X ≤ $(A_{FM} \vee B_{FM})_{FM}$ and $(A \vee B)_{FM} \leq (A_{FM} \vee B_{FM})_{FM}$. Conversely, $A_{FM} \leq (A \vee B)_{FM}$ and $B_{FM} \leq (A \vee B)_{FM}$ implies $(A_{FM} \vee B_{FM}) \leq (A \vee B)_{FM}$ and $(A_{FM} \vee B_{FM})_{FM} \leq [(A \vee B)_{FM}]_{FM} = (A \vee B)_{FM}$. Hence$(A_{FM} \vee B_{FM})_{FM} = (A \vee B)_{FM}$.

Theorem (2.11) If F ⊆ G are filters and A ∈ M then $A_{FM} \leq A_{GM}$.

Proof:- Let X ≤ A_{FM}. Then rX ≤ A, for some r ∈ F ⊆ G. Hence X ≤ A_{GM} and we have $A_{FM} \leq A_{GM}$.

Theorem (2.12) If A ∈ M, where A = aI_M, a ∈ F then $(a^n I_M)_{FM} = A_{FM} = I_M$.

Proof:- We have a^n ≤ a, for all n ∈ Z+. So $a^n I_M \leq aI_M$ and $(a^n I_M)_{FM} \leq (aI_M)_{FM} = A_{FM}$. Let X ≤ $A_{FM} = (a^n I_M)_{FM}$. Hence rX ≤ A = aI_M for some r ∈ F. Then a^{n-1} rX ≤ $a^n I_M$ which implies X ≤ $(a^n I_M)_{FM}$. Therefore $(a^n I_M)_{FM} = A_{FM}$. We have (A : a) = ∨{X ∈ M_*| aX ≤ A, a ∈ F} = ∨{X ∈ M_*| aX ≤ aI_M } = ∨{X ∈ M_*| X ≤ I_M} = $I_M \leq A_{FM}$.

Theorem (2.13) If p is a prime elements of L, where L is local with unique maximal element m, such that p ≤ m then $A_m \leq A_p$.

Proof:- A_m = ∨{xI_M | x ∈ L_*, yxI_M ≤ A, y ≰ m}, m being maximal is prime and p ≤ m. Let xI_M ≤ A_m, x ∈ L_*, yxI_M ≤ A, y ≰ m . Hence xI_M ≤ A_p and $A_m \leq A_p$

Theorem (2.14) Let p be minimal prime over a and m be a prime element in L. Then p ≤ m if and only if $A_m \leq A_p$ where A = aI_M ∈ M_*.

Proof:- If p ≤ m then $A_m \leq A_p$. Let Am ≤ A_p. Hence ∨{xI_M | x ∈ L_*, yxI_M ≤ A, y ≰ m} ≤ ∨ {xI_M | x ∈ L_*,yxI_M ≤ A, y ≰ p}. If xI_M ≤ A_m then yxI_M ≤ A and y ≰ m, for some y ∈ L which shows that y ≰ p. In particular for any y ≰ m, yA ≤ A, A ∈ M_* implies y ≰ p which shows that p ≤ m.

Theorem (2.15) If p and m are prime elements of L, such that A ≤ pI_M ≤ mI_M then $A_m \leq A_p \leq$ pI_M.

Proof:- We have, A_p = ∨{xI_M| x ∈ L_*, yxI_M ≤ A, y ≰ p}. Now pI_M ≤ mI_M implies p ≤ m and $A_m \leq A_p$. Also A = aI_M implies 1.aI_M ≤ A, 1 ≰ p. Hence aI_M = A ≤ A_p. Let xI_M ≤ A_p. Then yxI_M ≤ A, y ≰ p. But yxI_M ≤ pI_M, implies yx ≤ p where y ≰ p and p is prime. Hence x ≤ p so that xI_M ≤ pI_M. Thus A_p ≤ pI_M.

Theorem (2.16) If F and G are filters in F(L_*) then $A_{FM} \wedge G_{FM} = A_{(F \cap G)_M}$.

Proof:- Since F ⊆ F ⋂G, $A_{FM} \leq A_{(F \cap G)_M}$. Let X ∈ M_* such that X ≤ $A_{(F \cap G)_M}$.Thus rX ≤ A, for some r ∈ F ⋂G. Hence X ≤ A_{FM}, X ≤ A_{GM} and X ≤ $A_{FM} \wedge A_{GM}$, so $A_{(F \cap G)_M} \leq A_{FM} \wedge A_{GM}$. Conversely, let X be a compact element of M_* such that X ≤ $A_{FM} \wedge A_{GM}$. Then X ≤ A_{FM}, X ≤ A_{GM}, rX ≤ A, sX ≤ A, for some r, s ∈ F. Therefore (r ≤ s)X = (rX) ≤ (sX) ≤ A, where r ≤ s ∈ F. This shows that X ≤ $A_{(F \cap G)_M}$ and $A_{FM} \wedge A_{GM} \leq A_{(F \cap G)_M}$. Thus $A_{FM} \wedge A_{GM} = A_{(F \cap G)_M}$.

Theorem (2.17) Let p, q be prime elements of L such that (p ∧ q) is prime. Then $A_p \vee A_q \leq A_{(p\wedge q)}$.
Proof:- Let X ≤ A_p, X = xI_M, x ∈ L_* .Then yxI_M ≤ A, for some y ≰ p and hence x$I_M \leq A_{(p\wedge q)}$. This implies that $A_p \leq A_{(p\wedge q)}$. Similarly $A_q \leq A_{(p\wedge q)}$ and we have $A_p \vee A_q \leq A_{(p\wedge q)}$.
Theorem (2.18) $[\sqrt{(A : I_M)}]_p = \sqrt{(A : I_M)_p}$
Proof:- Let x ∈ L_* such that x ≤ $\sqrt{(A : I_M)_p}$. Then $x^n \leq (A : I_M)_p$ and x^nt ≤ (A : I_M) for some n ∈Z+, t ≰ p. This implies $x^n t^n = (xt)^n \leq x^n$t ≤ (A : I_M) i.e. xt ≤ $\sqrt{(A : I_M)}$, t ≰ p and we have x ≤ $[\sqrt{(A : I_M)}]_p$.
Conversely, let x ∈ L_* , x ≤ $\sqrt{(A : I_M)_p}$. Then xt ≤ $\sqrt{(A : I_M)}$, for some t ≰ p. So $x^n t^n = (xt)^n \leq$ (A : I_M).
Hence $x^n t^n$ ≤ A : I_M, t^n≰ p and x ≤ $\sqrt{(A : I_M)_p}$. Therefore, $[\sqrt{(A : I_M)}]_p = \sqrt{(A : I_M)_p}$.
Theorem (2.19) If A is a radical element i.e. (A : I_M) =$\sqrt{(A : I_M)}$ then, $(A : I_M)_p$ is a radical element.
Proof:- $(A : I_M)_p = [\sqrt{(A : I_M)}]_p = \sqrt{(A : I_M)_p}$. Hence $(A : I_M)_p$ is a radical element.
Theorem (2.20) If p is a prime element of L then $(pI_M)_p$ = pI_M.
Proof:- Let x$I_M \leq (pI_M)_p$, x ∈ L_* such that yxI_M ≤ pI_M, y ≰ p. Therefore yx ≤ p, y ≰ p implies x ≤ p and xI_M≤ pI_M. So $(pI_M)_p$ ≤ (pI_M). Conversely, let X ∈ M_* and X = xI_M ≤ pI_M, where x ∈ L_*. But 1.xI_M ≤ pI_M , 1 ≰ p implies x$I_M \leq (pI_M)_p$. Therefore, (pI_M) ≤ $(pI_M)_p$ and $(pI_M)_p$ = pI_M.
Theorem (2.21) $(I_M)_{FM} = I_M$.
Proof:- $(I_M)_{FM}$ = ∨{X ∈ M_* | sX ≤ I_M , for any s ∈ F} = I_M.
Theorem (2.22) If A is compact in M, p is prime in L and m is a unique maximal element of a local lattice L then $(A_m)_p = A_p$.
Proof:- Obviously A ≤ A_m , since 1.A ≤ A, 1 ≰ m. This implies $A_p \leq (A_m)_p$. Let Z = z$I_M \leq (A_m)_p$, z ∈ L_*. Then yz$I_M \leq A_m$; y ≰ p. This implies xyzI_M ≤ A, for some x ≰ m. But p ≤ m implies x ≰ p and hence xy ≰ p. Thus xyzI_M ≤ A, xy ≰ p implies zI_M ≤ Ap. Hence $(A_m)_p \leq A_p$ and we have $(A_m)_p = A_p$.
Theorem (2.23) Let G be a filter, p ∈ L be prime such that G ⊆F_p, then $V_p = (V_{GM})_p$.
Proof:- We have V_{GM}= ∨{X ∈ M_* | sX ≤ V, s ∈ G}, V_p = ∨{xI_M | x ∈ L_*, yxI_M ≤ V, y ≰ p} and F_p = {x ∈ L_*, x ≰ p}. Since 1V ≤ V, 1 ∈ G, we have V ≤ V_{GM} and $V_p \leq (V_{GM})_p$. Let xIM ≤ $(V_{GM})_p$, x ∈ L_*. Then yx$I_M \leq V_{GM}$, for some y ≤ p, and hence s(yxI_M) ≤ V , for some s ∈ G . This shows that s ∈ F_p, s ≰ p, y ≰ p and so sy ≰ p. Thus syxI_M ≤ V implies x$I_M \leq V_p$ and $(V_{GM})_p \leq V_p$. Therefore $V_p = (V_{GM})_p$.
Theorem (2.24) Let a multiplicative lattice L be local and m be a maximal element of L and p prime element of L such that p ≤ m. Then $(pI_M)_m$ = pI_M.
Proof:- $(pI_M)_m$ = ∨{xI_M |x ∈ L_*, yxI_M ≤ pI_M, y ≤ m}. We have ypI_M ≤ pI_M, for y ≰ m and hence p$I_M \leq (pI_M)_m$. Conversely, let x$I_M \leq (pI_M)_m$, x ∈ L_*. Thus yxI_M ≤ pI_M, for some y ≰m so that y ≰ p, as p ≤ m. This gives yx ≤ p, but p is prime implies x ≤ p. Hence xI_M ≤ pI_M and we have $(pI_M)_M$ = pI_M.
Theorem (2.25) If A = aI_M ∈ M_* then $(aI_M)_{FM} = I_M$ if and only if a ∈ F.
Proof:- Assume that, $(aI_M)_{FM} = I_M$. We have $(aI_M)_{FM}$ = ∨{X ∈ M_* | sX ≤ aI_M, for some s ∈ F} = I_M = ∨{(A : x) | x ∈ F}. (Theorem 2.3 [7]). Then I_M = {$X_1 \vee X_2 \vee ... \vee X_n$ | $s_i X_i$ ≤ aI_M, s_i ∈ F} or I_M = {(A : x_1)∨(A : x_2)∨ ...∨(A : x_n)}, since I_M is compact. But {(A : x_1)∨ (A : x_2)∨ ... ∨(A : x_n)} ≤ A : ($x_1 x_2 \ldots x_n$) [7] implies I_M ≤ A : $x_1 x_2 \ldots x_n$). This implies $(x_1 x_2 \ldots x_n) I_M$ ≤ A where x = $(x_1 x_2 \ldots x_n)$ ∈ F, for some x ∈ F which shows that x ≤ a; a ∈ F. Conversely, suppose a ∈ F. As a ∈ F, aI_M ≤ aI_M, we have $I_M \leq (aI_M)_{FM}$ and hence $(aI_M)_{FM} = I_M$.
Theorem (2.26) Let p be a prime element of L and A ∈ M_* . Then $A_p = I_M$ if and only if a ≰ p where A = aI_M for some a ∈ L.
Proof:- We have A = aI_M, a ∈ L. Assume that $A_p = I_M$. A_p = ∨{xI_M | x ∈ L_*, yxI_M ≤ A, y≰ p} = I_M. Now $I_M \leq (aI_M)_p$ implies yI_M ≤ aI_M, for some y≰ p. But y ≤ a gives a ≰ p. Conversely, let a≰ p. As 1 . aI_M≤ aI_M, a ≰ p we have $I_M \leq A_p$ and hence $A_p = I_M$.
Theorem (2.27) For V ∈ M, $V_{FM} \neq I_M$ is a $*_{V_M}$- element, where $*_{V_M}$ = {V_{FM} | F ∈ F(L_*), F ∩ [0, (V : I_M)] = ∅}.

Proof:- Suppose $V_{FM} \neq I_M$. But V_{FM} = ∨{X ∈ M_* | sX ≤ V; for some s ∈ F} $\neq I_M$ implies s$I_M \nleq$ V for any s ∈ F. As (V : I_M) I_M ≤ V implies (V : I_M) ∉ F, we have F ∩ [0, (V : I_M)] = ∅ . Hence $V_{FM} \neq I_M$ is a $*_V$ - element.

Theorem (2.28) If p is a prime element of L and A ∈ M then A ≤ pI_M if and only if $A_p \leq$ pI_M.

Proof:- Let $A_p \leq$ pI_M. Let X = xI_M ≤ A, x ∈ L_*. Then yxI_M ≤ A, y ≤ p. Therefore x$I_M \leq A_p \leq$ pI_M and hence A v pI_M. Conversely, let A ≤ pI_M. Let x ∈ L_*, such that x$I_M \leq A_p$. Then yxI_M ≤ A, y ≤ p. Thus yxI_M ≤ A ≤ pI_M. Therefore yx ≤ p where y ≤ p. Hence x ≤ p and $A_p \leq$ pI_M.

Theorem (2.29) If p is prime element of L then A ≤ pI_M if and only if $A_p \neq I_M$.

Proof:- Assume that $A_p \neq I_M$, we prove that A ≤ pI_M. Suppose that A≰ pI_M. Then there exists X ∈ M_* such that X = xI_M ≤ A but X ≰ pI_M i.e.xI_M ≤ pI_M and hence x ≰ p. This implies X = xI_M = 1.xI_M ≤ A, x ≰ p. So I_M = A_p , a contradiction and hence A ≤ pI_M. Conversely suppose A ≤ pI_M and Ap = I_M. Since I_M is compact , A_p = I_M = [f(A : x_1) ∨ (A : x_2) ∨… ∨ (A : x_n)}, $x_i \nleq$ p; i = 1, 2, 3…n] ≤ (A : $x_1 x_2 \ldots x_n$). As p is prime , $x_1 x_2 \ldots x_n \nleq$ p. ($x_1 \nleq$ p, $x_2 \nleq$ p… $x_n \nleq$ p) and 1.($x_1 x_2 \ldots x_n$) I_M ≤ A ≤ pI_M and hence $x_1, x_2 \ldots , x_n \leq$ p....... a contradiction. Hence $A_p \neq I_M$.

Theorem (2.30) Let p be a prime element of L then p is minimal prime over (A : I_M) if p = $(A : I_M)_p$. Converse holds if A : I_M is a radical element.

Proof:- Suppose p = $(A : I_M)_p$. Let x ≤ p = $(A : I_M)_p$. Then there exist y ∈ L_* such that yx ≤ (A : I_M), y ≰ p. Hence p is minimal prime over (A : I_M) by Theorem 3.3 of [8]. Conversely suppose p is a minimal prime over (A : I_M). We have (A : I_M) ≤ $(A : I_M)_p$ and (A : I_M) ≤ p. Let x ∈ L_* such that x ≤ $(A : I_M)_p$ = ∨{x ∈ L_* | xy ≤ (A : I_M); y ≰ p}. Then xt ≤ (A : I_M), t ≰ p. Hence xt ≤ p, t ≰ p implies x ≤ p. This shows that $(A : I_M)_p$ ≤ p. Let x ≤ p, then (Theorem 3.3 of [8]) there exist y ∈ L_*, y ≰ p such that x^ny ≤ (A : I_M), so x^n≤ $(A : I_M)_p$. This implies that x ≤ $\sqrt{(A : I_M)_p}$ = $[\sqrt{(A : I_M)}]_p$ = $\sqrt{(A : I_M)_p}$. (By Theorems 2.18, 2.19). Thus p ≤ $(A : I_M)_p$ and $(A : I_M)_p$ = p.

3. SIGMA ELEMENTS

An element A ∈ M is called σ-element if for any x ∈ L_*, xI_M ≤ A implies A ∨ (0_M : x) = I_M.
We write, σ_M = {A ∈ M | A is a σ - element}.

Theorem (3.1) Let M be a distributive multiplication Lattice module over a multiplicative lattice L and let A, B ∈ σ_M. Then A∧B ∈ σ_M and abI_M ∈ σ_M, where A = aI_M,B = bI_M.

Proof:- i) Let A, B be σ_M −elements and let x ∈ L_* be such that xI_M ≤ A ∧ B so xI_M ≤ A and xI_M ≤ B. Then A ∨(0_M : x) = I_M and B ∨(0_M : x) = I_M .But (A∧B) ∨(0_M : x)= [A∨ (0_M : x)] ∧ [B ∨(0_M : x)] = I_M ∧ I_M= I_M. Hence A ∧ B is σ_M-element of M.

ii) Let x ∈ L_* be such that xI_M ≤ abI_M. Then x ≤ ab, so x ≤ a, x ≤ b and hence xI_M ≤ aI_M and xI_M ≤ bI_M.As aI_M and bI_M are σ_M-elements, aI_M ∨ (0_M : x) = I_M and bI_M ∨ (0_M : x) = I_M. Since M is a multiplication lattice module and (0_M : x) ∈ M there exists d ∈ L such that (0_M : x) = dI_M. We have, aI_M ∨ dI_M = I_M and bI_M ∨ dI_M = I_M. This gives, (a ∨ d) I_M = I_M and (b ∨ d) I_M = I_M. But (a ∨ d) = 1 and (b ∨ d) = 1 implies ab ∨ d = 1 (By Theorem 2.14 of [4]).So abI_M∨ dI_M = abI_M∨(0_M : x) = I_M. Hence abI_M is σ -element.

Theorem (3.2) Let A,B ∈ σ_M. Then (A ∨ B) ∈ σ_M.

Proof:- Let x ∈ L_* be such that xI_M ≤ (A ∨ B).Since M be a multiplication Lattice module, A = aI_M,B = bI_M; a; b ∈ L. Then (A ∨ B) = (a ∨ b) I_M and xI_M ≤ (a ∨ b) I_M, so x ∨ (a ∨ b).Since x ∈ L_* and L is compactly generated there exist compact elements y, z ∈ L such that y ≤ a, z ≤ b, x ≤ y ∨ z. As A, B ∈ σ_M , yI_M ≤ A, zI_M ≤ B, we have, A ∨ (0_M : y) = I_M, B ∨ (0_M : z) = I_M. Then (A ∨ B) ∨ [(0_M : y) ∧ (0_M : z)] = (A ∨ B) ∨ [(0_M : y ∨ z)] [5]. Since M is distributive, (A ∨ B) ∨ [(0_M : y) ∧ (0_M : z)] = [(A ∨ B) ∨ (0_M : y)] ∧[(A ∨ B) ∨ (0_M : z)] = I_M. Hence (A ∨ B) ∨ [(0_M : y ∨ z)] = I_M. But x ∨ (y ∨ z) gives, [0_M : (y ∨ z)] ∨ (0_M : x). Hence [0_M : (y ∨ z)] ∨ (0_M : x) where x ∈ L_*, xI_M ∨ (A ∨ B). Hence (A ∨ B) ∈ σ_M.

Theorem (3.3) If $A_i \in \sigma_M$ then $\bigwedge_{i=1}^{n} A \in \sigma_M$.

Proof:-Let x $\in L_*$ be such that x$I_M \leq \wedge_{i\in\Delta}(A_i)$.Then x$I_M \leq A_i$ and $A_i \vee (0_M : x) = I_M$. Hence $(\wedge_i(A_i)) \vee (0_M : x) = [A_1 \vee (0_M : x)] \wedge [A_2 \vee (0_M : x)] \wedge \ldots \wedge [A_n \vee (0_M : x)] = I_M \wedge I_M \wedge ::: \wedge I_M = I_M$.

Theorem (3.4) If $A_\alpha \in \sigma_M$ then $\vee_\alpha A_\alpha \in \sigma_M$.

Proof:- Let x $\in L_*$ be such that x$I_M \leq \vee_\alpha(A_\alpha)$. As x$I_M$ is compact, x$I_M \leq A_1 \vee A_2 \vee \ldots \vee A_n$. Since M is compactly generated each A_i is the join of compact elements. Hence x$I_M \leq Y_1 \vee Y_2 \vee \ldots \vee Y_n$, for some $Y_i \in M_*$ such that $Y_i \leq A_i$, i = 1, 2, … n. As $Y_i \in M_*$ then there exist $y_i \in L_*$ such that $Y_i = y_i I_M$, 1 ≤ i ≤ n, $y_i I_M \leq A_i$. As each A_i is σ - element, $y_i \in L_*$, we have , $A_i \vee (0_M : y_i) = I_M$, i = 1,2,…, n, but A = $\vee_\alpha A_\alpha$implies A $\vee(0_M : y_i) = I_M$. Hence A $\vee [(0_M : y_1) \wedge (0_M : y_2) \wedge ::: \wedge (0_M : y_n)] = I_M$. So A$\vee [0_M : (y_1 \vee y_2 \vee \ldots \vee y_n)] = I_M$.
(By (vi) of [5]). If y = $(y_1 \vee y_2 \vee \ldots \vee y_n)$, then A $\vee (0_M : y) = I_M$. We have, x$I_M \leq y_1 I_M \vee y_2 I_M \vee \ldots \vee y_n I_M = (y_1 \vee y_2 \vee \ldots \vee y_n) I_M$. This gives x $\leq (y_1 \vee y_2 \vee \ldots \vee y_n)$ = y $\in L_*$. So 0_M: y $\leq 0_M$: x, x $\in L_*$ and A $\vee (0_M : x) = I_M$, where x $\in L_*$, x$I_M \leq \vee_\alpha A_\alpha$ = A. Therefore, A = $\vee_\alpha A_\alpha$ is a σ_M- element of M.

Theorem (3.5) 0_M and I_M are σ_M- elements of M.

Proof:- Let x $\in L_*$. Then obviously x$I_M \leq I_M$ and $I_M \vee (0_M : x) = I_M$. So I_M is σ_M- elements of M. If x $\in L_*$,then obviously x$I_M \leq 0_M$ gives x = 0 as M is torsion free, $0_M \vee (0_M : 0) = I_M$. Which shows that 0_M is σ_M- elements of M.

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