Integer solutions of the non homogeneous heptic equation with five unknowns $x^3 - y^3 - (x^2 + y^2) + z^3 - w^3 = 2 + 5(x - y)(z - w)^2 p^4$

Manju Somanath[*],V.Sangeetha[**] and M.A.Gopalan[***]

[*] Department of Mathematics, National College,Trichy-620001, Tamilnadu, India.
[**] Department of Mathematics, National College,Trichy-620001, Tamilnadu, India.
[***] Department of Mathematics, Srimathi Indira Gandhi College,Trichy-620002, Tamilnadu, India

Abstract- The non-homogeneous Diophantine equation of degree seven with five unknowns represented by $x^3 + y^3 - (x^2 + y^2) + z^3 - w^3 = 2 + 5(x - y)(z - w)^2 p^4$ is analyzed for its non – zero distinct integer solutions. Employing suitable linear transformations and applying the method of factorization, two different patterns of non-zero distinct integer solutions to the heptic equation under consideration are obtained. A few interesting relation between the solutions and special numbers are exhibited.

Index Terms- Non-homogeneous heptic, heptic equation with five unknowns, Integral solutions, special numbers.

2010 Mathematics Subject Classification: 11D 41

I. Introduction

Diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity as can be seen from[1,17].The problem of finding all integer solutions of a Diophantine equation with three or more variables and degree atleast three, in general presents a good deal of difficulties. There is vast general theory of homogeneous quadratic equations with three variables [2,3,4].Cubic equations in three variables fall into the theory of elliptic curves which is a very developed theory but still an important topic of current research [5-7].A lot is known about equations in two variables in higher degrees. For equations with more than three variables and degree at least three very little is known. It is worth to note that undesirability appears in equations, even perhaps at degree four with fairly small co-efficients. It seems that much work has not been done in solving higher order Diophantine equations. In [8-16], a few higher order equations are considered for integral solutions. In this communication a seventh degree non-homogeneous equation with five variables represented by $x^3 + y^3 - (x^2 + y^2) + z^3 - w^3 = 2 + 5(x - y)(z - w)^2 p^4$ is considered and in a few interesting relations among the solutions are presented.

Notations

- $t_{m,n}$ - Polygonal number of rank n with size m.
- P_n^m - Pyramidal number of rank n with size m.
- CP_n^m - Centered pyramidal number of rank n with size m.

II. METHOD OF ANALYSIS

The Diophantine equation representing the heptic equation with five unknowns under consideration is

$$x^3 + y^3 - (x^2 + y^2) + z^3 - w^3 = 2 + 5(x - y)(z - w)^2 p^4 \quad (1)$$

Introduction of the linear transformation

$$x = u + 1 \ ; \ y = u - 1 \ ; \ z = v + 1 \ ; \ w = v - 1 \quad (2)$$

in (1) leads to

$$4u^2 + 6v^2 = 40p^4 \quad (3)$$

Now, we solve (3) through two different methods to obtain different patterns of integer solutions to (1)

Pattern - I

Assume $p = p(a, b) = 4a^2 + 6b^2$ (4)

where *a* and *b* are non - zero distinct integers.

Write 40 as $40 = (4 + i2\sqrt{6})(4 - i2\sqrt{6})$ (5)

Using (4) & (5) in (3) and applying the method of factorization, define

$$2u + i\sqrt{6}v = \left(4 + i2\sqrt{6}\right)\left(2a + i\sqrt{6}b\right)^4$$

Equating the real and imaginary parts, we have

$$u = u(a, b) = 32a^4 - 192a^3 b - 288a^2 b^2 + 288a b^3 + 72b^4$$

$$v = v(a, b) = 32a^4 + 128a^3 b - 288a^2 b^2 - 192a b^3 + 72b^4$$

Hence in view of (2), the corresponding solutions of (1) are given by

$x = x(a,b) = 32a^4 - 192a^3b - 288a^2b^2 + 288ab^3 + 72b^4 + 1$

$y = y(a,b) = 32a^4 - 192a^3b - 288a^2b^2 + 288ab^3 + 72b^4 - 1$

$z = z(a,b) = 32a^4 + 128a^3b - 288a^2b^2 - 192ab^3 + 72b^4 + 1$

$w = w(a,b) = 32a^4 + 128a^3b - 288a^2b^2 - 192ab^3 + 72b^4 - 1$

$p = p(a,b) = 4a^2 + 6b^2$

A few interesting properties observed are as follows:

1. $x(1,b) - z(1,b) = 320CP_b^{18} + 160CP_b^6 - 320CP_b^{12}$.
2. $z(a,1) - w(a,1)$ is a kynea prime.
3. $p(a,1) - 6$ is a perfect square.
4. $4x(a,1) + 4y(a,1) + 1536CP_a^6 + 4608t_{3,a-1}$ is a sum of bi quadratic and square integer.
5. $6p(1,b)$ is a sum of nasty number and a square integer.

Note

In addition to (5), one may write 40 as $40 = (-4 + i2\sqrt{6})(-4 - i2\sqrt{6})$. For this choice, one obtains different patterns of solutions to (1).

Pattern-II

Rewrite (3) as $4u^2 + 6v^2 = 40p^4 * 1$ (6)

Write 1 as $1 = \frac{(5+i2\sqrt{6})(5-i2\sqrt{6})}{49}$ (7)

Using (5) and (7) in (6) and following the procedure similar to pattern-I, the corresponding non-zero distinct integral solutions of (1) are found to be

$x = x(A,B) = -10976A^4 - 592704A^3B + 98784A^2B^2 + 889056AB^3 - 24696B^4 + 1$

$y = y(A,B) = -10976A^4 - 592704A^3B + 98784A^2B^2 + 889056AB^3 - 24696B^4 - 1$

$z = z(a,b) = 98784A^4 - 43904A^3B - 889056A^2B^2 + 65856AB^3 + 222264B^4 + 1$

$w = w(a,b) = 98784A^4 - 43904A^3B - 889056A^2B^2 + 65856AB^3 + 222264B^4 - 1$

$$p = p(a,b) = 196A^2 + 294B^2$$

A few interesting properties of the above solution is presented below:

1. $9x(A,1) + z(A,1) + 32269440P_{A-1}^3 \equiv 0 (mod\ 10)$.
2. $9x(A,1) + z(A,1) + 1613460P_A^4 - 18823840t_{3,A} + 2689120t_{3,A-1} = 10$.
3. $p(1,B) + 294t_{4,B}$ is a perfect square.
4. $z(1,B) + 9y(1,B) = 10756480p_{B-1}^3 + 2689120CP_B^6 - 8$.
5. $9y(1,B) + z(1,B) = 5378240CP_B^{18} + 2689120CP_B^6 - 5378240CP_B^{12} - 8$.

Note

It is to be noted that one may write 1 as $1 = \frac{(-5+i2\sqrt{6})(-5-i2\sqrt{6})}{49}$. For this choice, one obtains different patterns of solutions to (1).

III. CONCLUSION

To conclude, one may search for other patterns of solutions and their corresponding properties to heptic equations, homogeneous or non-homogeneous with variables ≥ 5.

REFERENCES

[1] Dickson.L.E, “History of Theory of Numbers”, Vol.2, Chelsea Publishing Company, New York, 1952

[2] Gopalan.M.A ,Vidhalakshmi.S and Devibala.S, “Integral solutions of 49x^2+50y^2=51z^2”,Acta Ciencia Indica,XXXIIM(2), 2006,pp.839-840.

[3] Gopalan.M.A and Sangeetha.V and Manju Somanath, “Integral points on the homogeneous cone z^2=5x^2+11y^2”,Discovery Science, 3(7),2013,pp.05-08.

[4] Gopalan.M.A, “Note on the Diophantine equation x^2+xy+y^2=3z^2”,Acta Ciencia Indica,XXXVIM(3),2000,pp.265-266.

[5] Gopalan.M.A and Sangeetha.G, “On the ternary cubic Diophantine equation y^2=Dx^2+z^3”, Archimedes Journal of Mathematics,1(1),2010,pp.7-14.

[6] Gopalan.M.A,Manju Somanath and Vanitha.N , “Ternary Cubic Diophantine Equation x^2+y^2=2z^3”, Advances in Theoretical and Applied Mathematics,1(3),2006,2pp.27-231.

[7] Gopalan.M.A and Manju Somanath and Vanitha.N, “Ternary cubic Diophantine Equation 2^(2α-1) (x^2+y^2)=z^3”, ActacienciaIndica,XXXIVM(3),208,pp.1135-1137.

[8] Gopalan.M.A and Sangeetha.G, “Integral solutions of ternary non homogeneous biquadratic equation x^4+x^2+y^2-y=z^2+z”,ActaCienciaIndica,XXXVIIM(4),2011,pp.799-803.

[9] Gopalan.M.A, Manju Somanath and Sangeetha.G , Integral solutions of non homogeneous quartic equation x^4-y^4=(k^2+1)(z^2-w^2), Archimedes Journal of Mathematics,1(1),2011,pp.51-57.

[10] Gopalan.M.A and Sangeetha.G , “Integral solutions of Ternary quintic Diophantine equation x^2+y^2=(k^2+1) z^5”, Bulletin of pure and applied sciences, 29E (1),2010,pp.23-28.

[11] Gopalan.M.A and Janaki.G, "Integral solutions of (x^2-y^2)(3x^2+3y^2-2xy)=2(z^2-w^2)", Impact.J.sci.Tech, 4(1),2010,pp. 97-102.

[12] Gopalan.M.A and Vijayashankar.A , "Integral solutions of ternary quintic Diophantine equation x^2+(2k+1) y^2=z^5", International Journal of Mathematical Sciences 19(1-2), 2010,pp.165-169.

[13] Gopalan.M.A and Sangeetha.G, "On the sextic Diophantine equation with three unknowns X^2-XY+Y^2=(k^2+3)^n z^6",Impact.J.sci.Tech.,4(4),2010,pp.89-93.

[14] Gopalan.M.A and Sangeetha.G, "On the Heptic Diophantine equation with five unknowns x^4-y^4=(X^2-Y^2) z^5",Antarctica J.Math., 9(5),2012,pp.371-375.

[15] Manjula.M., Gopalan.M.A. and Vidhyalakshmi.S, "On the non-homogeneous heptic equation with five unknowns 〖(x〗^2-y^2)(4x^2+4y^2-6xy)=11(X^2-Y^2) z^5",Proceedings of ICOMAC, 2014,pp.275-278.

[16] Manju Somanath, Sangeetha.V and Gopalan.M.A., "On the non-homogeneous heptic equation with five unknowns 〖(x〗^2-y^2)(4x^2+4y^2-6xy)=8(X^2-Y^2) z^5". To be published.

[17] Mordell.L.J , Diophantine equations, Academic Press, London,1969.

AUTHORS

First Author Manju Somanath, Assistant Professor, National college,Trichy-1,e mail:manjuajil@yahoo.com
Second Author –Sangeetha.V,Assistant Professor, National college,Trichy-1,e mail:prasansangee@gmail.com
Third Author –Gopalan.M.A.,Professor,Srimathi Indira Gandhi College,Trichy-2,e mail:mayilgopalan@gmail.com

Correspondence Author –. Sangeetha.V ,e mail:prasansangee@gmail.com,phone:9790451555.