Double Dirichlet Average of M-series and Fractional Derivative

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Abstract- In this present paper we establish the results of Double Dirichlet average of M-Series and use fractional derivative. Some particular cases of our result are the generalization of earlier results.

Index Terms- Dirichlet average, M-Series, fractional calculus operators.

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I. Introduction

Carlson [1-5] has defined Dirichlet average of functions which represents certain type of integral average with respect to Dirichlet measure. He showed that various important special functions can be derived as Dirichlet averages for the ordinary simple functions like x^t, e^x etc. He has also pointed out [3] that the hidden symmetry of all special functions which provided their various transformations can be obtained by averaging x^n, e^x etc. Thus he established a unique process towards the unification of special functions by averaging a limited number of ordinary functions. Almost all known special functions and their well known properties have been derived by this process.

Recently, Gupta and Agarwal [9, 10] found that averaging process is not altogether new but directly connected with the old theory of fractional derivative. Carlson overlooked this connection whereas he has applied fractional derivative in so many cases during his entire work. Deora and Banerji [6] have found the double Dirichlet average of e^x by using fractional derivatives and they have also found the Triple Dirichlet Average of x^t by using fractional derivatives [7].

In the present paper the Dirichlet average of M-Series has been obtained.

II. Definitions

Some of the definitions which are necessary in the preparation of this paper.

1.1 Standard Simplex in $R^n, n \geq 1$:

Denote the standard simplex in R^n, $n \geq 1$ by [1, p.62].

$$E = E_n = \{S(u_1, u_2, \ldots \ldots .. u_n) : u_1 \geq 0, \ldots \ldots \ldots . u_n \geq 0, \; u_1 + u_2 + \cdots \ldots \ldots + u_n \leq 1\}$$

1.2 Dirichlet measure:

Let $b \in C^k, k \geq 2$ and let $E = E_{k-1}$ be the standard simplex in R^{k-1}. The complex measure μ_b is defined by $E[1]$.

$$d\mu_b(u) = \frac{1}{B(b)} u_1^{b_1-1} \ldots \ldots \ldots \ldots \ldots . u_{k-1}^{b_{k-1}-1}(1 - u_1 - \cdots \ldots \ldots \ldots - u_{k-1}) b_k^{-1} du_1 \ldots \ldots \ldots \ldots . du_{k-1}$$

knows as Dirichlet measure.

Here

$$B(b) = B(b1, \ldots \ldots \ldots . bk) = \frac{\Gamma(b_1) \ldots \ldots \ldots \ldots \ldots . \Gamma(b_k)}{\Gamma(b_1 + \cdots \ldots \ldots . . + b_k)},$$

$$C_> = \{z \in z: z \neq 0, |ph\ z| < \pi/2\},$$

Open right half plane and $C_>$k is the k^{th} Cartesian power of $C_>$

1.3 Dirichlet Average[1, p.75]:

Let Ω be the convex set in $C_>$, let $z = (z_1, \dots \dots \dots, z_k) \in \Omega^k, k \geq 2$ and let $u.z$ be a convex combination of $z_1, \dots \dots \dots, z_k$. Let f be a measureable function on Ω and let μ_b be a Dirichlet measure on the standard simplex E in R^{k-1}.Define

$$F(b,z) = \int_E f(u.z) d\,\mu_b(u) \qquad (2.3)$$

F is the Dirichlet measure of f with variables

$z = (z_1, \dots \dots \dots, z_k)$ and parameters $b = (b_1, \dots \dots \dots . b_k)$.

Here

$$u.z = \sum_{i=1}^{k} u_i z_i \text{ and } u_k = 1 - u_1 - \cdots \dots \dots . -u_{k-1.}$$

If $k = 1$, define $F(b,z) = f(z).$

The following notation have been used in present work,

$R^k = k^{th}$, Cartesian product of $C_>$,
$R =$ Set of real numbers,
$C_> =$ Open right half plane,
$\mu_b =$ Complex measure,
$\Omega^k = k^{th}$, Cartesian product of Ω
$\Omega =$ Convex set in $C_>$,
$B(b) =$ Beta function
$E =$ Standard simplex

1.4 Fractional Derivative [8, p.181]:

The theory of fractional derivative with respect to an arbitrary function has been used by Erdelyi[8]. The most common definition for the fractional derivative of order α found in the literature on the "Riemann-Liouville integral" is

$$D_z^\alpha F(z) = \frac{1}{\Gamma(-\alpha)} \int_0^z F(t) (z-t)^{-\alpha-1} dt \qquad (2.4)$$

Where $Re(\alpha) < 0$ and $F(x)$ is the form of $x^p f(x)$, where $f(x)$ is analytic at $x = 0.$

2.5 Average of $\cosh x$ (from [4]):

let μ^b be a Dirichlet measure on the standard simplex E in R^{k-1}; k$\geq$ 2. For every $z \in C^k$

$$S(b,z) = \int_E {}_pM_q^\alpha(u,z) d\mu_b\ (u) \qquad (2.5)$$

If $k = 1, S = (b,z) = {}_pM_q^\alpha(u,z)$.

2.6 Double averages of functions of one variable (from [1, 2]):

let z be a $k \times x$ matrix with complex elements z_{ij}. Let $u = (u_1, \dots \dots \dots u_k) and\ v = (v_1, \dots \dots \dots \dots v_k)$ be an ordered k-tuple and x-tuple of real non-negative weights $\sum u_i = 1\ and\ \sum v_j = 1,$ respectively.
Define

$$u.z.v = \sum_{i=1}^{k}\sum_{j=1}^{x} u_i z_{ij} v_j \qquad (2.6)$$

If z_{ij} is regarded as a point of the complex plane, all these convex combinations are points in the convex hull of $(z_{11}, \dots \dots \dots z_{kx})$, denote by $H(z)$.

Let $b = (b_1, \dots \dots \dots . b_k) be$ an ordered k -tuple of complex numbers with positive real part(Re(b) > 0) and similarly for β=$(\beta_{1, \dots\dots} \beta_x)$. Then we define $d\mu_b(u)$ and $d\mu_b(v)$.

Let f be the holomorphic on a domain D in the complex plane, If $Re(b) > 0, Re(\beta) > 0 and H(z) \subset D,$ define

$$F(b,z,\beta) = \iint f(u,z,v)\, d\mu_b(u) d\mu_b(v) \qquad (2.7)$$

Corresponding to the particular function $\mathrm{M} - \mathrm{Series}, z^t and\ e^z,$ define,

$$S(b,z,\beta) = \iint {}_pM_q^\alpha(u,z,v)\, d\mu_b(u)\, d\mu_b(v) \qquad (2.8)$$

$$R_t(b,z,\beta) = \iint (u,z,v)^t d\mu_b(u)\, d\mu_b(v) \qquad (2.9)$$

$$S(b,z,\beta) = \iint (e)^{u.z.v} d\mu_b(u)\, d\mu_b(v) \qquad (2.10)$$

III. Main Results and Proof

Theorem: Following equivalence relation for Double Dirichlet Average is established for ($k = x = 2$) of ${}_pM_q^\alpha(u,z,v)$

$$S(\mu,\mu';z;\rho,\rho') = \frac{\Gamma(\rho+\rho')}{\Gamma\rho}(x-y)^{1-\rho-\rho'} D_{x-y}^{-\rho'}\, {}_pM_q^\alpha(x)\,(x-y)^{\rho-1} \qquad (3.1)$$

Proof:

Let us consider the double average for $(k = x = 2$ of ${}_pM_q^\alpha(u,z,v)$

$$S(\mu,\mu';z;\rho,\rho') = \int_0^1\int_0^1 {}_pM_q^\alpha(u,z,v)\, dm_{\mu,\mu'}(u) dm_{\rho,\rho'}(v)$$

$$= \sum_{n=o}^{\infty} \frac{(a_1)_n \; . \; . \; . \; . \; . (a_p)_n}{(b_1)_n \; . \; . \; . \; . \; . (b_q)_n} \frac{1}{\Gamma(\alpha n+1)} \int_0^1\int_0^1 [u.z.v]^n dm_{\mu,\mu'}(u) dm_{\rho,\rho'}(v)$$

$Re(\mu) = 0, Re(\mu') = 0, Re(\rho) > 0, Re(\rho') > 0$ and

$$u.z.v = \sum_{i=1}^{2}\sum_{i=1}^{2}(u_i z_{ij} v_j) = \sum_{i=1}^{2}[u_i(z_{i1}v_1 + z_{i2}v_2)]$$

$$= [u_1 z_{11} v_1 + u_1 z_{12} v_2 + u_2 z_{21} v_1 + u_2 z_{22} v_2]$$

let $z_{11} = a, z_{12} = b, z_{21} = c, z_{22} = d$ and $\begin{cases} u_1 = u, & u_2 = 1-u \\ v_1 = v, & v_2 = 1-v \end{cases}$

Thus $z = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$v.z.v = uva + ub(1-v) + (1-u)cv + (1-u)d(1-v)$$

$$= uv(a-b-c+d) + u(b-d) + v(c-d) + d$$

$$dm_{\mu,\mu'}(u) = \frac{\Gamma(\mu+\mu')}{\Gamma\mu\,\Gamma\mu'} u^{\mu-1}(1-u)^{\mu'-1}du$$

$$dm_{\rho,\rho'}(v) = \frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'} v^{\rho-1}(1-v)^{\rho'-1}dv$$

Putting these values in ( ), we have,

$$S(\mu,\mu';z;\rho,\rho') = \frac{\Gamma(\mu+\mu')}{\Gamma\mu\,\Gamma\mu'}\frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'} \times \sum_{n=o}^{\infty} \frac{(a_1)_n \;.\;.\;.\;.\;.(a_p)_n}{(b_1)_n \;.\;.\;.\;.\;.(b_q)_n} \frac{1}{\Gamma(\alpha n+1)} \times$$

$$\int_0^1\int_0^1 [uv(a-b-c+d) + u(b-d) + v(c-d) + d]^n\, u^{\mu-1}(1-u)^{\mu'-1} v^{\rho-1}(1-v)^{\rho'-1}dudv$$

In order to obtained the fractional derivative equivalent to the above integral, we assume $a = c = x; b = d = y$ then

$$S(\mu,\mu';z;\rho,\rho') = \frac{\Gamma(\mu+\mu')}{\Gamma\mu\,\Gamma\mu'}\frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'} \times \sum_{n=o}^{\infty} \frac{(a_1)_n \;.\;.\;.\;.\;.(a_p)_n}{(b_1)_n \;.\;.\;.\;.\;.(b_q)_n} \frac{1}{\Gamma(\alpha n+1)}$$

$$\times \int_0^1\int_0^1 [uv(x-y)+y]^n\, u^{\mu-1}(1-u)^{\mu'-1} v^{\rho-1}(1-v)^{\rho'-1}dudv$$

$$S(\mu,\mu';z;\rho,\rho') = \frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'} \times \sum_{n=o}^{\infty} \frac{(a_1)_n \;.\;.\;.\;.\;.(a_p)_n}{(b_1)_n \;.\;.\;.\;.\;.(b_q)_n} \frac{1}{\Gamma(\alpha n+1)} \int_0^1 [uv(x-y)+y]^n\, v^{\rho-1}(1-v)^{\rho'-1}dv$$

Putting $v(x-y) = t,$ we obtain

$$S(\mu,\mu';z;\rho,\rho') = \frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'} \times \sum_{n=o}^{\infty} \frac{(a_1)_n \;.\;.\;.\;.\;.(a_p)_n}{(b_1)_n \;.\;.\;.\;.\;.(b_q)_n} \frac{1}{\Gamma(\alpha n+1)}$$

$$\times \int_0^{x-y} [y+t]^n \left(\frac{t}{x-y}\right)^{\rho-1} \left(1-\frac{t}{x-y}\right)^{\rho'-1} \frac{dt}{(x-y)}$$

$$= \frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}(x-y)^{1-\rho-\rho'} \sum_{n=o}^{\infty} \frac{(a_1)_n \;.\;.\;.\;.\;.(a_p)_n}{(b_1)_n \;.\;.\;.\;.\;.(b_q)_n} \frac{1}{\Gamma(\alpha n+1)} \int_0^{x-y} [y+t]^n\,(t)^{\rho-1}(x-y-t)^{\rho'-1}dt$$

On changing the order of integration and summation, we have

$$S(\mu,\mu';z;\rho,\rho') = \frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}(x-y)^{1-\rho-\rho'} \int_0^{x-y} {}_pM_q^{\alpha}(y+t)\,(t)^{\rho-1}(x-y-t)^{\rho'-1}dt$$

Using definition of fractional derivative (2.4), we get

$$S(\mu,\mu';z;\rho,\rho') = \frac{\Gamma(\rho+\rho')}{\Gamma\rho}(x-y)^{1-\rho-\rho'} D_{x-y\,p}^{-\rho'} M_q^{\alpha}(x)\,(x-y)^{\rho-1}$$

This is complete proof of (3.1).

Particular cases:

(i) From Ali, Jain and Sharma, we get

(ii)

$$S(\beta,\beta',x,y) = \frac{\Gamma(\beta+\beta')}{\Gamma\beta}(x-y)^{1-\beta-\beta'} D_{x-y\,p}^{-\beta'} M_q^{\alpha}(x)(x-y)^{\beta-1}$$

Thus

$$S(\beta,\beta',x,y) = S(\mu,\mu';z;\rho,\rho')$$

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