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On The Homogeneous Bi-quadratic Equation with FiveUnknowns $x^4 - y^4 = 8(z+w)p^3$

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Abstract-The biquadratic equation with 5 unknown given by $x^4 - y^4 = 8(z+w)p^3$ is analyzed for its patterns of non – zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.

Index Terms- Bi-quadratic equation with 5 unknowns, Homogeneous biquadratic, Integer solutions, Special polygonal numbers, Centered polygonal number

I. Introduction

Biquadratic diophantine equations, homogeneous and non- homogeneous, have aroused the interest of numerous Mathematicians since antiquity as can be seen from [1-7]. In the context one may refer [8-24] for varieties of problems on the diophantine equations with two, three and four variables. This communication concerns with the problem of determining non-zero integral solutions of yet another biquadratic equation in 5 unknowns represented by $x^4 - y^4 = 8(z+w)p^3$.A few interesting relations between the solutions and special polygonal numbers are presented.

II. Notations

$t_{m,n}$ - Polygonal number of rank n with size m.

$Ct_{m,n}$ - Centered polygonal number of rank n with size m.

gn_a - Gnomonic number of rank a

SO_n - Stella octangular number of rank n

$CP_{m,n}$ - Centered pyramidal number of rank n with size m

III. Method of Analysis

The diophantine equation representing the biquadratic equation with five unknowns under consideration is

$$x^4 - y^4 = 8(z+w)p^3 \quad (1)$$

The substitution of the transformations

$$x = u+v,\ y = u-v,\ z = ruv+1,\ w = suv-1 \quad (2)$$

in (1) leads to $u^2 + v^2 = (r+s)p^3$ (3)

To solve (3), we apply the method of factorization. For this, choose r and s such that r+sis expressed as the product of complex conjugates. A few illustrations are presented below.

A. Illustration 1

Assume $p = a^2 + b^2$ (4)

Take r=7 , s=3

so that $r+s = 10 = (3+i)(3-i)$ (5)

Using (4) & (5) in (3) and employing the method of factorization, define

$$(u+iv) = (3+i)(a+ib)^3$$

Equating the real and imaginary parts, we have

$$u = u(a,b) = 3a^3 - 9ab^2 - 3a^2b + b^3$$
$$v = v(a,b) = a^3 - 3ab^2 + 9a^2b - 3b^3$$

Hence in view of (2), the corresponding solutions of (1) are given by

$$x = x(a,b) = 4a^3 - 12ab^2 + 6a^2b - 2b^3$$
$$y = y(a,b) = 2a^3 - 6ab^2 - 12a^2b + 4b^3$$
$$z = z(a,b) = 7[(3a^3 - 9ab^2 - 3a^2b + b^3)(a^3 - 3ab^2 + 9a^2b - 3b^3)] + 1$$
$$w = w(a,b) = 3[(3a^3 - 9ab^2 - 3a^2b + b^3)(a^3 - 3ab^2 + 9a^2b - 3b^3)] - 1$$
$$p = p(a,b) = a^2 + b^2$$

A few interesting properties observed are as follows:

1. $2x(a,a) - y(a,a) - 5so_a \equiv 0 (\bmod 25)$
2. $z(a,b) - w(a,b) = x^2(a,b) - y^2(a,b) + 2$
3. $z(a,b) + w(a,b) = \frac{5}{2}[x^2(a,b) - y^2(a,b)]$
4. $30[x(a,a) - 2y(a,a) + p(a,a) - 2t_{4,a}]$ is a nasty number:
5. $50\{x(a,a) - 2y(a,a)\}$ is a cubical integer.

B. Illustration 2

The choice r=20 , s=5

so that $r + s = 25 = (1+2i)^2(1-2i)^2$

Following a similar procedure as in illustration-1,the corresponding solutions of (3) are found to be

$$u = u(a,b) = -3a^3 + 9ab^2 - 12a^2b + 4b^3$$
$$v = v(a,b) = 4a^3 - 12ab^2 - 9a^2b + 3b^3$$

Hence in view of (2), the corresponding solutions of (1) are given by

$$x = x(a,b) = a^3 - 3ab^2 - 21a^2b + 7b^3$$
$$y = y(a,b) = -7a^3 + 21ab^2 - 3a^2b + b^3$$
$$z = z(a,b) = 20[(-3a^3 + 9ab^2 - 12a^2b + 4b^3)(4a^3 - 12ab^2 - 9a^2b + 3b^3)] + 1$$
$$w = w(a,b) = 3[(-3a^3 + 9ab^2 - 12a^2b + 4b^3)(4a^3 - 12ab^2 - 9a^2b + 3b^3)] - 1$$
$$p = p(a,b) = a^2 + b^2$$

A few interesting properties observed are as follows:

1. $7x(a,b) + y(a,b) \equiv 0 (\bmod 50b)$
2. $z(a,b) - w(a,b) - 4[x^2(a,b) - y^2(a,b)] \equiv 2 (\bmod uv)$
3. $z(a,b) - 4w(a,b) \equiv 0 (\bmod 5)$
4. $-7x(a,1) - y(a,1) + 50$ isa nasty number.

5. $20\{7x(1,b)+y(1,b)+75(gn_b-1)\}$ is a cubical integer.

C. Illustration 3

Let r=6 , s=2

so that $r+s=8=(1+i)^3(1-i)^3$

Following a similar procedure as in illustration-2, the corresponding solutions of (3) are as follows

$$u=u(a,b)=-2a^3+6ab^2-6a^2b+2b^3$$
$$v=v(a,b)=2a^3-6ab^2-6a^2b+2b^3$$

Hence in view of (2), the corresponding solutions of (1) are given by

$$x=x(a,b)=4b^3-12a^2b$$
$$y=y(a,b)=-4a^3+12ab^2$$
$$z=z(a,b)=6[(2a^3-6a^2b)^2-(2a^3-6ab^2)^2]+1$$
$$w=w(a,b)=2[(2a^3-6a^2b)^2-(2a^3-6ab^2)^2]-1$$
$$p=p(a,b)=a^2+b^2$$

A few interesting properties observed are as follows:

1. $x(a,a)+y(a,a)=0$
2. $2z(a,b)=2+3[x^2(a,b)-y^2(a,b)]$
3. $2w(a,b)=-2+x^2-y^2$
4. $3\{z(a,b)+w(a,b)+2y^2(a,b)\}$ is a nasty number:
5. $4\{y(a,a)-x(a,a)\}$ is a cubical integer.

D. Illustration 4

Let r=13 , s=3

so that $r+s=16=(1+i)^4(1-i)^4$

Following a similar procedure as in illustration-1, the corresponding solutions of (3) are as follows

$$u=u(a,b)=-4a^3+12ab^2$$
$$v=v(a,b)=-12a^2b+4b^3$$

Hence in view of (2), the corresponding non-zero integral solutions of (1) are given by

$$x=x(a,b)=-4a^3+12ab^2-12a^2b+4b^3$$
$$y=y(a,b)=-4a^3+12ab^2+12a^2b-4b^3$$
$$z=z(a,b)=13(12ab^2-4a^3)(4b^3-12a^2b)+1$$
$$w=w(a,b)=3(12ab^2-4a^3)(4b^3-12a^2b)-1$$
$$p=p(a,b)=a^2+b^2$$

A few interesting properties observed are as follows:

1. $x(a,a)+y(a,a)+4so_a=10(gn_a-1)$

2. $2[z(a,b)-w(a,b)=4+5[x^2(a,b)-y^2(a,b)]$

3. $z(a,b)+w(a,b)=-4(x^2(a,b)-y^2(a,b))$

4. $2p(a,a)$ isa nasty number.

5. $-x(a,a)+y(a,a)+3CP_{3,a}$ is a cubical integer.

E. Illustration 5

Take r=1 , s=3

so that $r+s=4=\dfrac{(1+i)^{2n+2}(1-i)^{2n+2}}{2^{2n}}$

Employing the method of factorization, define

$$(u+iv)=\frac{(1+i)^{2n+2}}{2^n}(a+ib)^3$$

$$(u+iv)=2[\cos(2n+2)\frac{\pi}{4}+i\sin(2n+2)\frac{\pi}{4}](a+ib)^3$$

Equating the real and imaginary parts, we have

$$u=u(a,b)=2[(a^3-3ab^2)\cos(2n+2)\frac{\pi}{4}-(3a^2b-b^3)\sin(2n+2)\frac{\pi}{4}]$$

$$v=v(a,b)=2[(a^3-3ab^2)\sin(2n+2)\frac{\pi}{4}-(3a^2b-b^3)\cos(2n+2)\frac{\pi}{4}]$$

Hence in view of (2), the corresponding solutions of (1) are given by

$$x=x(a,b)=2\{(a^3-3ab^2)[\cos(2n+2)\frac{\pi}{4}+\sin(2n+2)\frac{\pi}{4}]+(3a^2b-b^3)[\cos(2n+2)\frac{\pi}{4}-\sin(2n+2)\frac{\pi}{4}]\}$$

$$y=y(a,b)=2\{(a^3-3ab^2)[\cos(2n+2)\frac{\pi}{4}-\sin(2n+2)\frac{\pi}{4}]-(3a^2b-b^3)[\cos(2n+2)\frac{\pi}{4}+\sin(2n+2)\frac{\pi}{4}]\}$$

$$z=z(a,b)=4\{(a^3-3ab^2)^2[\cos(2n+2)\frac{\pi}{4}\sin(2n+2)\frac{\pi}{4}]+(a^3-3ab^2)(3a^2b-b^3)*$$

$$[\cos^2(2n+2)\frac{\pi}{4}-\sin^2(2n+2)\frac{\pi}{4}]-(3a^2b-b^3)^2[\cos(2n+2)\frac{\pi}{4}\sin(2n+2)\frac{\pi}{4}]\}+1$$

$$w=w(a,b)=12\{(a^3-3ab^2)^2[\cos(2n+2)\frac{\pi}{4}\sin(2n+2)\frac{\pi}{4}]+(a^3-3ab^2)(3a^2b-b^3)*$$

$$[\cos^2(2n+2)\frac{\pi}{4}-\sin^2(2n+2)\frac{\pi}{4}]-(3a^2b-b^3)^2[\cos(2n+2)\frac{\pi}{4}\sin(2n+2)\frac{\pi}{4}]\}-1$$

$$p=p(a,b)=a^2+b^2$$

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